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On the perturbative chiral ring for marginally deformed $\mathcal{N} = 4$ SYM theories

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Abstract: For $\mathcal{N} = 1$ $SU(N)$ SYM theories obtained as marginal deformations of the $\mathcal{N} = 4$ parent theory we study perturbatively some sectors of the chiral ring in the weak coupling regime and for finite N. By exploiting the relation between the definition of chiral ring and the effective superpotential we develop a procedure which allows us to easily determine protected chiral operators up to n loops once the superpotential has been computed up to $(n-1)$ order. In particular, for the Lunin–Maldacena β–deformed theory we determine the quantum structure of a large class of operators up to three loops. We extend our procedure to more general Leigh–Strassler deformations whose chiral ring is not fully understood yet and determine the weight–two and weight–three sectors up to two loops. We use our results to infer general properties of the chiral ring.

Keywords: Supersymmetric gauge theory, Superspaces, AdS-CFT Correspondence.

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1. Introduction

The original formulation of the AdS/CFT correspondence [1 – 3] involves a SYM theory with maximal supersymmetry. First steps in the direction of studying the correspondence with a lower number of supersymmetries were undertaken in [4].

If $\mathcal{N} = 1$ superconformal invariance is required the field theory can be realized by orbifold constructions [5] or by the *exactly* marginal deformations of the $\mathcal{N} = 4$ SYM first classified in [6]. The second class of theories has been extensively studied in a field theory approach [6 – 9] and in the context of the AdS/CFT correspondence [10 – 14].

The interest in marginal deformed SYM theories has recently received a considerable boost thanks to the work of Lunin–Maldacena [15] where the gravity dual of the so called β–deformed theory has been proposed. It corresponds to the low energy limit of a string theory on a deformed background $\mathrm{AdS}_5 \times \mathrm{S}^5_\beta$ obtained by SL(2,R) transforming the τ

modulus of a two–torus inside S^5. Alternatively, it can be obtained from the original $AdS_5 \times S^5$ solution by applying a TsT transformation in S^5 [15 – 18].

A considerable effort has been devoted so far to provide tests of the correspondence in its marginal deformed version. As for the $AdS_5 \times S^5$ original correspondence, perturbative properties of the field theory have been investigated: For the $SU(N)$ case the condition which constrains the couplings of the theory in order to have $\mathcal{N} = 1$ superconformal invariance has been determined perturbatively up to three loops [19 – 21]. In the large N limit the exact superconformal condition has been found in [22]. Nonrenormalization properties of operators in the chiral ring have been established perturbatively [19 – 21] and multiloop amplitudes have been computed [20, 21, 23]. The exact anomalous dimensions for spin–2 operators of the form $\mathrm{Tr}(\Phi_1^J \Phi_2)$ have been determined [22] for N, J unrelated and large[1]. Finally, the gauge one–loop effective action has been computed [25] for a particular background configuration. Nonperturbative instantonic effects have been also considered [26].

Integrability properties of the original $\mathcal{N} = 4$ SYM theory (see [27] for a review and list of references) survive the β–deformation [28, 29, 16, 30, 31] and Bethe ansatz techniques can be used also in this case to compute the spectrum of anomalous dimensions of composite operators.

On the string theory side BPS states have been investigated in [32] for orbifold configurations. Integrability properties have been exploited on the two sides of the correspondence in order to match the energies of semiclassical fast rotating strings with one–loop anomalous dimensions of scalar operators [33 – 37]. The spectrum of states has been also investigated in the BMN limit [38, 39].

Non–supersymmetric generalizations of the Lunin–Maldacena β–deformation have been proposed [16] and further investigations have been carried on [40, 17, 41 – 43]. Very recently, deformations obtained by acting with TsT transformations in AdS_5 have been also proposed [44].

Finally, the Lunin–Maldacena deformation has been applied in the context of dipole theories with the purpose of disentangling the KK modes (whose dynamics gets affected by the deformation) from the gauge modes [45 – 49].

In a previous paper [20] we have initiated the study of the chiral ring of the $SU(N)$ β–deformed SYM theory by exploiting perturbative techniques in $\mathcal{N} = 1$ superspace [50 – 54]. There we concentrated on the single–trace sector of the chiral ring: For the lowest dimensional scalar operators we proved the vanishing of their anomalous dimensions up to two loops and the appearance of finite corrections to their correlation functions, in contradistinction to the $\mathcal{N} = 4$ case. In particular, our two–loop results confirmed the protection [19] of the operator $\mathrm{Tr}(\Phi_i \Phi_j)$, $i \neq j$ which was missing in the list of CPO's of the theory [10, 11, 15].

In this paper we intend to pursue our investigation and extend it to higher dimensional sectors of the chiral ring for scalar chiral superfields. We work at finite N and take into

[1]The same kind of limit has been recently considered in [24] for studying magnons in the $\mathcal{N} = 4$ SYM theory.

account mixing among sectors with different trace structures. Exploiting the definition of quantum chiral ring we reduce the determination of protected operators up to order n in perturbation theory to the evaluation of the effective superpotential up to order $(n-1)$. Precisely, from the knowledge of the effective superpotential we determine perturbatively all the quantum descendant operators of naive scale dimension Δ_0, and find the CPO's as the operators which are orthogonal order by order to the descendants.

For the β–deformed theory we investigate the spin–2 sector[2] and applying our procedure to simple cases ($\Delta_0 = 4, 5$) we determine the protected operators up to three loops. In the sectors we have studied we can always define descendant operators which do not receive quantum corrections. This seems to be a general property of the spin–2 operators: Despite the nontrivial appearance of finite perturbative corrections to the effective action, the quantum descendant operators defined in terms of the effective superpotential coincide with their expressions given in terms of the classical superpotential (up to possible mixing among them).

We then investigate the spin–3 sector where, due to the appearance of Konishi–like anomalies, we need restrict our analysis at two loops in order to avoid dealing with mixed gauge/scalar operators. Up to this order the descendant operators we consider are the classical ones. However, in this sector we expect higher order corrections to the descendants to appear together with a nontrivial dependence on the anomaly term. Therefore, the non–renormalization properties of the descendant operators that we experiment for the spin–2 sector are not a general feature of the theory.

We generalize our procedure to the study of protected operators for the $\mathcal{N} = 1$ superconformal theory associated to the full Leigh–Strassler deformation. Even if the gravity dual of this theory is not known yet, it is anyway interesting to figure out the general structure of its chiral ring. Still at finite N, we study explicitly the weight–2 and weight–3 sectors up to two loops and perform a preliminary analysis of the general sectors at least at lowest order in the couplings. An interesting result we find is that, because of the discrete Z_3 symmetries of the theory, the sectors corresponding to conformal weights which are multiple of 3 have a different operator structure from the other ones.

The plan of the paper is as follows: After an introductory section on the β–deformed superconformal theory, in section 3 we introduce the definition of perturbative chiral ring and discuss the general procedure we adopt to determine the CPO's of the theory. In section 4 we compute the perturbative effective superpotential up to two loops as required to determine protected operators up to three loops. These are then the subject of sections 5 and 6 for the spin–2 and spin–3 sectors, respectively. In section 7 we study the more general $\mathcal{N} = 1$ superconformal theory described by the full Leigh–Strassler superpotential. Some conclusions follow plus an appendix on loop integrals we used in our calculations.

2. Generalities on the β–deformed theory

Given the $\mathcal{N} = 4$ SYM theory in $\mathcal{N} = 1$ superspace notation we consider its deformation [6,

[2] We use the notation of [33] and call "spin–n" the sector containing operators made by products of n different flavors.

15]

$$
\begin{aligned}
S = & \int d^8z \ \mathrm{Tr}\left(e^{-gV}\overline{\Phi}_i e^{gV}\Phi^i\right) + \frac{1}{2g^2}\int d^6z \ \mathrm{Tr}W^\alpha W_\alpha \\
& + ih\int d^6z \ \mathrm{Tr}(q\,\Phi_1\Phi_2\Phi_3 - \bar{q}\,\Phi_1\Phi_3\Phi_2) - i\bar{h}\int d^6\bar{z} \ \mathrm{Tr}(\bar{q}\,\overline{\Phi}_1\overline{\Phi}_3\overline{\Phi}_2 - q\,\overline{\Phi}_1\overline{\Phi}_2\overline{\Phi}_3)
\end{aligned}
\tag{2.1}
$$

where we have set $q \equiv e^{i\pi\beta}$, $\bar{q} \equiv e^{-i\pi\beta}$, β real. The gauge coupling g has been chosen to be real in order to avoid dealing with instantonic effects, whereas h is generically complex.

The superfield strength $W_\alpha = i\overline{D}^2(e^{-gV}D_\alpha e^{gV})$ is given in terms of a real prepotential V and $\Phi_{1,2,3}$ contain the six scalars of the original $\mathcal{N}=4$ SYM theory organized into the $\mathbf{3}\times\bar{\mathbf{3}}$ of $SU(3)$ subgroup of the R–symmetry group $SU(4)$. We write $V = V^aT_a$, $\Phi_i = \Phi_i^aT_a$ where T_a are $SU(N)$ matrices in the fundamental representation[3].

The β–deformation breaks $\mathcal{N}=4$ supersymmetry to $\mathcal{N}=1$ and the original $SU(4)$ R–symmetry to $U(1)_R$. However, two extra non–R–symmetry global $U(1)$'s survive. Applying the a–maximization procedure [55] and the conditions of vanishing ABJ anomalies it turns out that $U(1)_R$ is the one which assigns the same R–charge ω to the three elementary superfields, whereas the charges with respect to the two non–R–symmetries $U(1)_1\times U(1)_2$ can be chosen to be $(\Phi_1,\Phi_2,\Phi_3)\to(0,1,-1)$ and $(-1,1,0)$, respectively.

The action (2.1) possesses two extra discrete symmetries. One is the Z_3 associated to cyclic permutations of (Φ_1,Φ_2,Φ_3) which is a remnant of the original $SU(3)\subset SU(4)$ symmetry of the undeformed theory, whereas the other one corresponds to exchanges

$$
\Phi_i \leftrightarrow \Phi_j\,, \quad i\neq j \qquad \text{and} \qquad q\to -\bar{q} \quad (\beta\to 1-\beta)
\tag{2.2}
$$

The equations of motion for the chiral superfields are

$$
\bar{D}^2(e^{-gV}\bar{\Phi}_1^a e^{gV}) = -ih\Phi_2^b\Phi_3^c\ [q(abc)-\bar{q}(acb)]
\tag{2.3}
$$

and cyclic, where $(abc)\equiv \mathrm{Tr}(T^aT^bT^c)$.

At the quantum level the theory is superconformal invariant (and then finite) up to two loops if the coupling constants satisfy the following condition (vanishing of the beta functions) [19, 20]

$$
|h|^2\left[1-\frac{1}{N^2}\,|q-\bar{q}|^2\right] = g^2
\tag{2.4}
$$

Superconformal invariance at three loops has been discussed in [21] for any N. In the large N limit this condition reduces simply to $|h|^2=g^2$, independently of the value of q. In [22] it has been proven that this is the *exact* superconformal invariance condition for the large N theory dual to the Lunin–Maldacena supergravity background [15].

In this paper we consider the $\mathcal{N}=1$ superconformal theory at *finite* N, perturbatively defined by the condition (2.4) and investigate at the quantum level some sectors of its chiral ring.

[3] For more details on our conventions we refer to [52–54, 20].

3. The chiral ring of the β-deformed theory

We are interested in studying perturbatively the structure of the chiral ring for the β–deformed theory (2.1). As discussed in [56], for a generic $\mathcal{N}=1$ SYM theory scalar operators in the chiral ring can be constructed as products of scalar chiral superfields Φ_i and/or times $(W^\alpha W_\alpha)$, where W_α is the chiral field strength. In this paper we will focus only on the Φ–sector, neglecting operators with a dependence on W_α.

In [10, 11, 15] the single–trace sector of the chiral ring has been identified as given by chiral operators of the form $\mathrm{Tr}(\Phi_1^{J_1}\Phi_2^{J_2}\Phi_3^{J_3})$ with weight $\Delta_0 = J_1+J_2+J_3$ and $(J_1,J_2,J_3) = (J,0,0),(0,J,0),(0,0,J),(J,J,J)$. In [19, 20] it has been shown perturbatively that also the assignements $(J_1,J_2,J_3) = (1,1,0),(1,0,1),(0,1,1)$ give protected operators.

This classification identifies the CPO's according to their dimension and their charges with respect to the two $U(1)$ global invariances of the theory. However, it does not give any information on the precise form of the protected operator corresponding to a given set (J_1,J_2,J_3), which turns out to be in general a linear combination of single–trace operators with different order of the fields inside the trace. Moreover, if we work at finite N, mixing with multi–trace operators is also allowed.

A first example has been studied in [19] for the weight–3 sector. There, it has been shown that the correct expression for the protected operator correponding to $(J_1,J_2,J_3) = (1,1,1)$ is a linear combination

$$\mathrm{Tr}(\Phi_1\Phi_2\Phi_3) + \alpha\mathrm{Tr}(\Phi_1\Phi_3\Phi_2) \tag{3.1}$$

where at one–loop

$$\alpha = \frac{(N^2-2)\bar{q}^2+2}{N^2-2+2\bar{q}^2} \tag{3.2}$$

showing an explicit dependence on the coupling β.

We are interested in the generalization of this result to higher loops in order to investigate whether and how the linear combination gets modified order by order. Moreover, we extend this analysis to other sectors of the chiral ring in order to discuss mixing at finite N.

In general, given a set of primary operators $\mathcal{O}_i$ with the same dimension Δ_0 and the same global charges, we can read their anomalous dimensions perturbatively from the matrix of the two–point correlation functions. Precisely, this matrix has the form

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle = \frac{1}{x^{2\Delta_0}}\left(A_{ij} - \rho_{ij}\log\mu^2x^2 \; + \; \cdots\right) \tag{3.3}$$

where dots stay for higher powers in $\log\mu^2x^2$. Here A is the mixing matrix, whereas ρ signals the appearance of anomalous dimensions. Both matrices are given as power series in the couplings.

In order to determine the anomalous dimensions we need diagonalize the two matrices by performing the linear transformation $\mathcal{O}' = L\mathcal{O}$ which maps the operators into an orthogonal basis of quasi–primaries. In a perturbative approach it is easy to see [57, 58] that the diagonalization of the ρ matrix at order n fixes the correct orthogonalization (resolution of

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the mixing) at order $(n-1)$ uniquely, up to a residual rotation among operators with the same anomalous dimension. This means that in general an order n calculation is required to determine the anomalous dimensions at this order and the correct linear combinations of operators $\mathcal{O}_i$ at order $(n-1)$ which correspond to quasi–primaries with well–defined anomalous dimensions up to order n.

In our case, since we are interested into **chiral primary operators**, the procedure to determine perturbatively the correct linear combination which corresponds to a protected operator is made simpler if we also use the definition of chiral ring.

In our conventions the chiral ring is the set of chiral operators which cannot be written, by using the equations of motion, as $\bar{D}^2 X$, being X any primary operator.

In general, given a set of linearly independent chiral operators $\mathcal{C}_i$, $i = 1, \cdots, s$ characterized by the same classical scale dimension Δ_0 and the same charges under the two $U(1)$ flavor groups they will mix and we need solve the mixing in order to compute their anomalous dimensions. Since we are working with chiral operators, we know a priori that once we have orthogonalized as $\mathcal{C}'_i = L_{ij}\mathcal{C}_j$ in order to have well–defined quasi–primary operators, some of them will turn out to be descendant, i.e. they can be written as $\bar{D}^2 X$ for some primary X. The remaining operators will be necessarily primary chirals with vanishing anomalous dimensions.

Exploiting this simple observation, in order to find the correct expression for the protected operators, we then proceed as follows: In a given (J_1, J_2, J_3) sector, we first select all the descendants, that is all the linear combinations

$$\mathcal{D}_i = \sum_j d_j^{(i)} \mathcal{C}_j \tag{3.4}$$

which satisfy the condition

$$\mathcal{D}_i = \bar{D}^2 X_i \tag{3.5}$$

Let us suppose that there are $i = 1, \ldots, r \leq s$ independent linear combinations of this type. Then, for a generic operator $\mathcal{P} = \sum_j c_j \mathcal{C}_j$ we impose the orthogonality condition

$$\langle \mathcal{P} \bar{\mathcal{D}}_i \rangle = 0 \qquad i = 1, \ldots, r \tag{3.6}$$

where $\bar{\mathcal{D}}$ indicates the hermitian conjugate of $\mathcal{D}$. These constraints provide r equations for the s unknowns c_j. In this way we select a $(s-r)$-dimensional subspace of operators orthogonal to the descendant ones. We can choose an appropriate (orthogonal) basis in this subset, obtaining $(s-r)$ independent operators which are protected. This procedure has been already applied in the undeformed $\mathcal{N} = 4$ case [59].

The problem of determining the CPO's of the theory is then traslated into the problem of finding *all* the linear combinations of operators which satisfy the condition (3.5). In particular, since we are interested into a perturbative determination of the chiral ring we need find descendants which solve eq. (3.5) order by order in perturbation theory. This can be done by introducing a perturbative definition of quantum chiral ring, as we are now going to explain in detail.

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3.1 The perturbative quantum chiral ring

As previously discussed, the chiral ring is defined as the set of chiral operators orthogonal to null operators, i.e. linear combinations of chirals which can be written in the form $\bar{D}^2 X$, X primary. At the classical level a linear combination (3.4) gives rise to a null operator every time the coefficients $d_j^{(i)}$ are such that the operator $\mathcal{D}_i$ can be rewritten as a product of chiral superfields times $\frac{\delta W}{\delta \Phi_k}$, where W is the classical superpotential[4]

$$W = ih \left[q \ \mathrm{Tr}(\Phi_1 \Phi_2 \Phi_3) - \bar{q} \ \mathrm{Tr}(\Phi_1 \Phi_3 \Phi_2) \right] \tag{3.7}$$

Indeed, if this is the case, we can use the classical equations of motion $\bar{D}^2 \bar{\Phi}_k = -\frac{\delta W}{\delta \Phi_k}$ to express the operator as in (3.5). It follows that we can alternatively define the chiral ring as

$$\mathcal{C} = \{ \text{chiral op.}'\text{s } \mathcal{P} \mid \langle \mathcal{P} \bar{\mathcal{D}} \rangle = 0, \ \text{for any } \mathcal{D} \sim (\ldots \Phi .. \Phi .. \frac{\delta W}{\delta \Phi}) \} \tag{3.8}$$

where in $\mathcal{D}$ we do not indicate trace structures and flavor charges explicitly. In the undeformed $\mathcal{N} = 4$ theory, an immediate consequence of the definition (3.8) is that all the CPO's correspond to completly symmetric representations of the $SU(3) \subset SU(4)$ R–symmetry group [3].

This definition for the chiral ring allows for a straightforward generalization at the quantum level. Since the quantum dynamics of the elementary superfields is driven by the effective superpotential rather than the classical W, it appears natural to define the quantum chiral ring as

$$\mathcal{C}_Q = \{ \text{chiral op.}'\text{s } \mathcal{P} \mid \langle \mathcal{P} \bar{\mathcal{D}}_Q \rangle = 0, \ \text{for any } \mathcal{D}_Q \sim (.. \Phi \ldots \Phi \ldots \frac{\delta W_{eff}}{\delta \Phi}) \} \tag{3.9}$$

where now $\mathcal{D}_Q$ is a *quantum* null operator. Using the quantum equations of motion $\bar{D}^2 \frac{\delta K}{\delta \Phi_i} = -\frac{\delta W_{eff}}{\delta \Phi_i}$ where K is the effective Kähler potential which takes into account possible perturbative D–term corrections, it is easy to see that $\mathcal{D}_Q$ is a null operator at the quantum level. In the undeformed $\mathcal{N} = 4$ case the symmetries of the theory constrain $\mathcal{D}_Q$ to be proportional to $\mathcal{D}$ and the quantum chiral ring coincides with the classical one (3.8).

When W_{eff} is determined perturbatively, eq. (3.9) gives a perturbative definition of chiral ring. Precisely, given W_{eff} at a fixed perturbative order[5]

$$W_{eff} = W + \lambda W_{eff}^{(1)} + \lambda^2 W_{eff}^{(2)} + \cdots + \lambda^L W_{eff}^{(L)} \tag{3.10}$$

we can construct independent descendants [6] at that order as

$$\mathcal{D} = \mathcal{D}_0 + \lambda \mathcal{D}_1 + \lambda^2 \mathcal{D}_2 + \cdots + \lambda^L \mathcal{D}_L \qquad , \qquad \mathcal{D}_i = \Phi \ldots \frac{\delta W_{eff}^{(i)}}{\delta \Phi} \tag{3.11}$$

[4]This is true only for operators which are not affected by Konishi–like anomalies or as long as these anomalies do not enter the actual calculation (see the discussion at the beginning of section 6).

[5]In principle, perturbative corrections to W_{eff} would depend on both g and h couplings. Here we mean to use the superconformal invariance condition to express $|h|^2$ as a function of g^2 and write the perturbative expansion in powers of the 't Hooft coupling $\lambda = \frac{g^2 N}{4\pi^2}$.

[6]As long as we are interested in orthogonalizing with respect to the whole space generated by the descendants, we do not need the precise form of pure descendants, but just a suitable set of linear independent states. From now on we will refer to this definition of quantum descendants.

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and determine the protected operators $\mathcal{P}$ by imposing the orthogonality condition $\langle \mathcal{P}\bar{\mathcal{D}}\rangle = 0$ order by order. Since $\mathcal{P}$ will be in general a linear combination of single/multitrace operators, these conditions allow to determine the coefficients of the linear combination order by order in the couplings. If we set

$$\mathcal{P} = \mathcal{P}_0 + \lambda\mathcal{P}_1 + \lambda^2\mathcal{P}_2 + \cdots + \lambda^L\mathcal{P}_L \tag{3.12}$$

the perturbative corrections $\mathcal{P}_j$ will be determined by

$$\begin{aligned} O(\lambda^0): \quad & \langle \mathcal{P}_0\bar{\mathcal{D}}_0\rangle_0 = 0 \\ O(\lambda^1): \quad & \langle \mathcal{P}_0\bar{\mathcal{D}}_1\rangle_0 + \langle \mathcal{P}_0\bar{\mathcal{D}}_0\rangle_1 + \langle \mathcal{P}_1\bar{\mathcal{D}}_0\rangle_0 = 0 \\ \vdots \quad & \quad \vdots \\ O(\lambda^L): \quad & \langle \mathcal{P}_0\bar{\mathcal{D}}_L\rangle_0 + \langle \mathcal{P}_0\bar{\mathcal{D}}_{L-1}\rangle_1 + \cdots + \langle \mathcal{P}_0\bar{\mathcal{D}}_0\rangle_L + \langle \mathcal{P}_1\bar{\mathcal{D}}_{L-1}\rangle_0 + \cdots + \langle \mathcal{P}_L\bar{\mathcal{D}}_0\rangle_0 = 0 \end{aligned} \tag{3.13}$$

where $\langle\ \rangle_j$ stands for the two–point function at order λ^j.

Conditions (3.13) together with the general statement that orthogonalization at order $(n-1)$ is sufficient for having well–defined quasi–primary operators at order n, brings us to formulate the following prescription: In order to determine perturbatively the correct form of chiral operators with vanishing anomalous dimension at order n it is sufficient to determine the effective superpotential at order $(n-1)$, select all the descendant operators at that order by (3.11) and impose the conditions (3.13) up to order $(n-1)$. In so doing, we gain a perturbative order at each step. Moreover, in order to have all the descendants at a given order it is sufficient to compute the effective superpotential once for all.

As follows from its definition, the structure of the chiral ring is directly related to the structure of the effective superpotential. Therefore, the perturbative corrections to the CPO's depend on the perturbative corrections to the effective superpotential. In particular, this explains universality properties of the protected operators we will discuss in section 5, as for example the fact that in any case the orthogonalization at tree level is sufficient for the protection up to two loops.

4. The effective superpotential at two–loops

Since we are dealing with a superconformal (finite) theory any correction to the effective action must be finite. By definition, the effective superpotential corresponds to perturbative, finite F–terms evaluated at zero momenta. It is given by *local* contributions which are constrained by dimensions, $U(1)\times U(1)$ flavor symmetry charges, reality and symmetry (2.2) to have necessarily the form

$$W_{eff} = ih\left[b\ \mathrm{Tr}(\Phi_1\Phi_2\Phi_3) - \bar{b}\ \mathrm{Tr}(\Phi_1\Phi_3\Phi_2)\right] \ + \ \text{h.c.} \tag{4.1}$$

The constant b is given as an expansion in the couplings, $b = q(1 + b_1\lambda + b_2\lambda^2 + \cdots)$, with coefficients b_j which are functions of q and N, whereas $\bar{b}$ is the hermitian conjugate. We note that in principle the symmetries of the theory would only constrain the form of the superpotential to $W_{eff} = \{ih\,[b(q)\ \mathrm{Tr}(\Phi_1\Phi_2\Phi_3) + b(-\bar{q})\ \mathrm{Tr}(\Phi_1\Phi_3\Phi_2)]\ + \ \text{h.c.}\}$. However,

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it is easy to show that $b(-\bar{q}) = -\overline{b(q)}$ since the b_j coefficients are rational functions of q^2 with real coefficients (loop diagrams always give real contributions and they always contain an even number of extra chiral vertices compared to the tree–level vertex).

At a given order L we can have two kinds of corrections to W_{eff}: Corrections which do not mix the two terms in the superpotential and are then of the form

$$W_{eff}^{(L)} \sim \lambda^L \, W \tag{4.2}$$

where W is the classical superpotential. These contributions do not affect the structure of the descendant operators at order L since $\frac{\delta W_{eff}^{(L)}}{\delta \Phi} \sim \frac{\delta W}{\delta \Phi}$ and $\mathcal{D}_L \sim \mathcal{D}_0$. As a consequence at order L the correlation function $\langle \mathcal{P}_0 \bar{\mathcal{D}}_L \rangle_0$ in (3.13) vanishes and the protected operator is determined only by loop corrections to its two–point function with descendants of lower orders.

The second kind of corrections to W_{eff} mixes the two terms in W and gives rise to a linear combination $W_{eff}^{(L)}$ of the form (4.1) which is not proportional to the classical superpotential anymore. For these corrections the request for the protected operator to be orthogonal to a descendant proportional to $\frac{\delta W_{eff}^{(L)}}{\delta \Phi}$ modifies in general its structure by contributions of order λ^L proportional to $\langle \mathcal{P}_0 \bar{\mathcal{D}}_L \rangle_0$.

In this section we evaluate explicitly the effective superpotential up to two loops. Our result is useful for determining the correct CPO's up to three loops.

The diagrams contributing to the effective superpotential up to this order are given in figure 1 where the grey bullets indicate the one–loop corrections to the chiral and gauge–chiral vertices, respectively. These corrections are exactly the ones of the undeformed $\mathcal{N} = 4$ theory once we use the one–loop superconformal invariance condition (2.4).

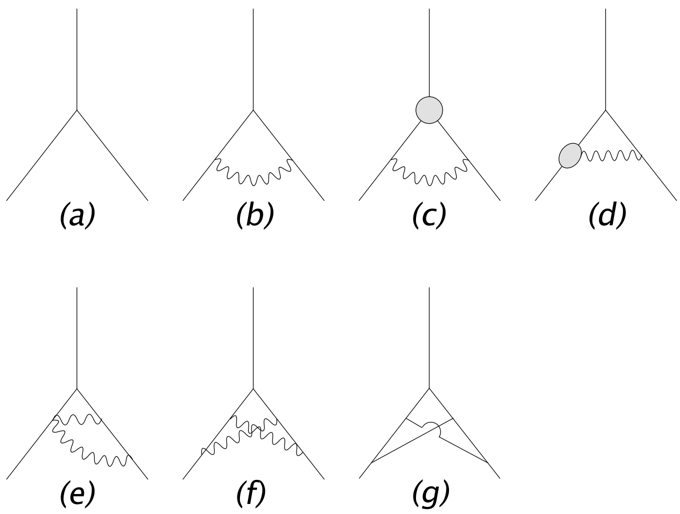

Figure 1: Diagrams contributing to the effective superpotential up to two loops.

The one–loop diagram 1b), compared with the tree level diagram 1a), does not contain any extra q–deformed vertex. Moreover, using standard color identities it is easy to see that its contribution is proportional to λW, where W is the classical superpotential.

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The same happens at two loops for the diagrams 1c), 1d) and 1e) which do not contain any extra q–deformed vertex and have a color structure which does not mix the two traces, so reproducing W.

Diagram 1f) vanishes for color reasons.

Diagram 1g) contains four extra q–deformed vertices. Moreover, by direct inspection one can easily see that the nonplanar chiral structure which corrects the tree level diagram mixes nontrivially the two terms of W. As a result at two loops the superpotential undergoes a nontrivial modification of the form

$$W^{(2)}_{eff} \sim ih \left[q\, P \ \mathrm{Tr}(\Phi_1 \Phi_2 \Phi_3) - \bar{q}\, \bar{P} \ \mathrm{Tr}(\Phi_1 \Phi_3 \Phi_2) \right] \ + \ \mathrm{h.c.} \tag{4.3}$$

with

$$P = \frac{(q^2-1)^3 [N^2 + 3 + q^2(3N^2 - 10 + 7q^2)]}{q^2[q^4 + 1 + (N^2-2)q^2]^2} \tag{4.4}$$

Here we have used $\bar{q} = 1/q$. We note that the nontrivial q–dependence of this diagram is a direct consequence of its nonplanarity. In fact, as discussed in [22] planar diagrams depend on the particular combination $q\bar{q} = 1$, while the nonplanar ones have generically nontrivial phases. Moreover, a q–dependence has also been introduced by using the superconformal condition (2.4) to express the coefficient $|h|^4$ from the four chiral vertices in terms of λ^2.

To evaluate the various contributions from figure 1 we first perfom D–algebra to reduce superdiagrams to ordinary loop diagrams and compute the corresponding integrals in momentum space (for the description of the procedure and our conventions we refer to [52–54, 20]). As reported in appendix A the one and two–loop integrals are all finite and they give a well–defined, local value for external momenta set to zero. Therefore, collecting all the contributions, at two loops the superpotential has the structure (4.1) with

$$b = q \left[(1 + \lambda c_1 + \lambda^2 c_2) + \lambda^2 \frac{3}{8} \zeta(3) P \right] \tag{4.5}$$

where the coefficients c_1, c_2 are numbers, independent of q and N, determined by the loop integrals 1b) and 1c)–1e), respectively (we do not need their explicit values).

It follows that in general a descendant at this order will have the form

$$\mathcal{D}_Q = (1 + \lambda c_1 + \lambda^2 c_2)\mathcal{D}_0 + \lambda^2 \mathcal{D}_2 \tag{4.6}$$

with $\mathcal{D}_2 \neq \mathcal{D}_0$.

5. Chiral Primary Operators in the spin–2 sector

5.1 The $(J, 1, 0)$ flavor

We start considering operators of the form $\mathrm{Tr}(\Phi_1^J \Phi_2)$. In this case, due to the ciclicity of the trace, there is no ambiguity in the ordering of the operators inside the trace. In the large N limit these operators do not belong to the chiral ring, they are descendants and their anomalous dimensions have been computed exactly [22] for J large. However, for finite N they can mix with multitraces and give rise to linear combinations of single and multi–trace operators which are protected. We are going to construct them perturbatively up to three loops. For simplicity we consider first the particular cases of $J = 3, 4$ and postpone the discussion for generic J at the end of this section.

The $(3,1,0)$ case: The first nontrivial example where mixing conspires to give rise to protected operators is for $J = 3$. This sector contains the two operators

$$\mathcal{O}_1 = \mathrm{Tr}(\Phi_1^3\Phi_2) \qquad , \qquad \mathcal{O}_2 = \mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_1\Phi_2) \tag{5.1}$$

Using the classical equations of motion (2.3), it is easy to see that

$$\bar{D}^2\mathrm{Tr}(\Phi_1^2 e^{-gV}\bar{\Phi}_3 e^{gV}) = \mathrm{Tr}\left(\Phi_1^2\frac{\delta W}{\delta\Phi_3}\right) = -ih\,(q-\bar{q})\,[\mathrm{Tr}(\Phi_1^3\Phi_2) - \frac{1}{N}\mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_1\Phi_2)] \tag{5.2}$$

and a descendant can be constructed as (we always forget about the normalization of the operators)

$$\mathcal{D}_0 = \mathcal{O}_1 - \frac{1}{N}\mathcal{O}_2 \tag{5.3}$$

The knowledge of $\mathcal{D}_0$ allows us to determine the one–loop protected operator. We consider the linear combination

$$\mathcal{P}_0 = \mathcal{O}_1 + \alpha_0\,\mathcal{O}_2 \tag{5.4}$$

which, for any $\alpha_0 \neq -\frac{1}{N}$, gives an operator in the chiral ring. We then impose the orthogonality condition $\langle\mathcal{P}_0\,\bar{\mathcal{D}}_0\rangle_0 = 0$ and find

$$\alpha_0 = -\frac{N^2-6}{2N} \tag{5.5}$$

This result coincides with the one found in [21] where the one–loop CPO has been determined by diagonalizing directly the one–loop anomalous dimension matrix.

In order to extend our analysis to higher loops we need establish the correct form of the descendant operator order by order, as described in section 3. If we look at its perturbative definition (3.11) and the way the equations of motion work in this case, we easily realize that as long as the effective superpotential has the structure (4.1) we obtain

$$\mathrm{Tr}\left(\Phi_1^2\frac{\delta W_{eff}}{\delta\Phi_3}\right) = -ih\,\left(b-\bar{b}\right)[\mathrm{Tr}(\Phi_1^3\Phi_2) - \frac{1}{N}\mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_1\Phi_2)] \tag{5.6}$$

whatever b might be (determined perturbatively at a given order). It follows that the linear combination on the r.h.s. of this equation, which is nothing but the operator (5.3), is always a descendant operator independently of the order we have computed the coefficient b. Therefore we conclude that (5.3) is the *exact* quantum descendant up to an overall coupling–dependent normalization factor, that is $\mathcal{D}_Q \sim \mathcal{D}_0$.

An alternative way [59] to establish the relation $\mathcal{D}_Q \sim \mathcal{D}_0$ is to consider the combination

$$\bar{D}^2\mathrm{Tr}(\Phi_1^2 e^{-gV}\bar{\Phi}_3 e^{gV}) + ih\,(q-\bar{q})\,[\mathrm{Tr}(\Phi_1^3\Phi_2) - \frac{1}{N}\mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_1\Phi_2)] \tag{5.7}$$

which is zero at tree level and check that it is order by order orthogonal to the three monomials $\bar{D}^2\mathrm{Tr}(\Phi_1^2 e^{-gV}\bar{\Phi}_3 e^{gV})$, $\mathrm{Tr}(\Phi_1^3\Phi_2)$ and $\mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_1\Phi_2)$, separately. In fact, if this

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is the case, there is no extra mixing of the linear combination (5.7) with the three operators at the quantum level and (5.7) must be necessarily zero at any order in perturbation theory. We have checked the absence of mixing perturbatively up to two loops confirming our conclusion.

In order to determine the protected operator we consider the linear combination

$$\mathcal{P} = \mathcal{O}_1 + \alpha\,\mathcal{O}_2 \tag{5.8}$$

with α given as an expansion in λ

$$\alpha = \alpha_0 + \alpha_1\,\lambda + \alpha_2\,\lambda^2 + O(\lambda^3) \tag{5.9}$$

In the notation of section 3 we have $\mathcal{P}_0 = \mathcal{O}_1 + \alpha_0\mathcal{O}_2$ with α_0 already determined in (5.5) and $\mathcal{P}_j = \alpha_j\mathcal{O}_2$.

As a consequence of the relation $\mathcal{D}_Q \sim \mathcal{D}_0$ the orthogonality conditions (3.13) become

$$O(\lambda)\ : \qquad \langle\mathcal{P}_0\bar{\mathcal{D}}_0\rangle_1 + \langle\mathcal{P}_1\bar{\mathcal{D}}_0\rangle_0 = 0 \tag{5.10}$$

$$O(\lambda^2) : \qquad \langle\mathcal{P}_0\bar{\mathcal{D}}_0\rangle_2 + \langle\mathcal{P}_1\bar{\mathcal{D}}_0\rangle_1 + \langle\mathcal{P}_2\bar{\mathcal{D}}_0\rangle_0 = 0 \tag{5.11}$$

The first condition (5.10) gives

$$\alpha_1 = -\frac{\langle(\mathcal{O}_1 + \alpha_0\mathcal{O}_2)\bar{\mathcal{D}}_0\rangle_1}{\langle\mathcal{O}_2\bar{\mathcal{D}}_0\rangle_0} \tag{5.12}$$

In order to select the diagrams which contribute to the two–point function at the numerator we note that the tree level correlation function at the denominator, when computed in momentum space and in dimensional regularization ($n = 4 - 2\epsilon$), is $1/\epsilon$ divergent. This divergence signals the well–known short distance singularity of any two–point function of a conformal field theory.

If the denominator of (5.12) goes as $1/\epsilon$, in the numerator we can consider only divergent diagrams (finite diagrams would not contribute in the $\epsilon \to 0$ limit). It is easy to show that at this order the only diagram which we need take into account is the one in figure 2 where on the left hand side we have an insertion of the operator $(\mathcal{O}_1 + \alpha_0\mathcal{O}_2)$ while on the right hand side we have $\bar{\mathcal{D}}_0$.

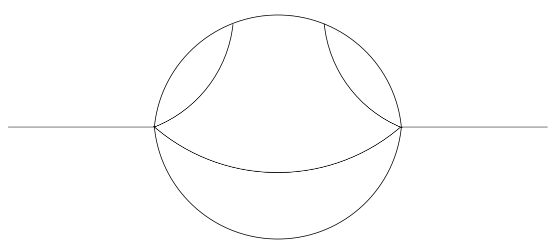

Figure 2: One–loop diagram contributing to the evaluation of α_1.

By a direct calculation one realizes that if α_0 is chosen as in (5.5) this diagram vanishes. The reason is very simple to understand: If we cut the diagram vertically at the very right end, close to the $\bar{\mathcal{D}}_0$ vertex, from the calculation it comes out that the left part would be nothing but a one–loop divergent contribution to the operator $(\mathcal{O}_1 + \alpha_0\mathcal{O}_2)$ which vanishes since α_0 has been determined just to give a protected (not renormalized) operator at one–loop.

From the one–loop constraint we then read $\alpha_1 = 0$ and the expression (5.4) with α_0 as in (5.5) corresponds to the protected chiral operator up to two loops.

Next we analyze the constraint (5.11). Setting $\mathcal{P}_1 = 0$ there, we obtain

$$\alpha_2 = -\frac{\langle(\mathcal{O}_1 + \alpha_0\mathcal{O}_2)\,\bar{\mathcal{D}}_0\rangle_2}{\langle\mathcal{O}_2\,\bar{\mathcal{D}}_0\rangle_0} \tag{5.13}$$

and consequently the exact expression for the CPO up to three loops.

Again we select only divergent diagrams contributing to the numerator. They are given in figure 3. We have not drawn diagrams associated to the two–loop anomalous dimension of the operator $(\mathcal{O}_1 + \alpha_0\mathcal{O}_2)$ which vanish when α_0 is chosen as in (5.5).

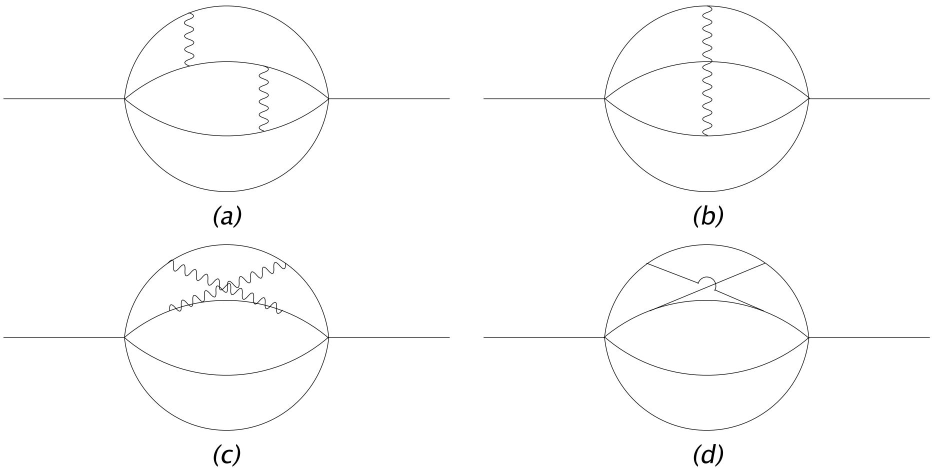

Figure 3: Two-loop diagrams contributing to the evaluation of α_2.

These diagrams contribute nontrivially to α_2 since, cutting the graphs at the very right hand side, their left parts cannot be recognized as corrections to the tree–level operator (nontrivial mixing between $\mathcal{O}_1$ and $\mathcal{O}_2$ occurs). Evaluating the diagrams by using the results in appendix A we obtain

$$\alpha_2 = \frac{9(N^2-9)(q^2-1)^2[(N^4-8N^2-8)(q^4+1)+2(N^4+8)q^2]}{80N[q^4+1+(N^2-2)q^2]^2}\,\zeta(3) \tag{5.14}$$

where we have used the one–loop superconformal condition (2.4) to express all the contributions of figure 3 in terms of λ^2 and set $\bar{q} = 1/q$.

Therefore the protected operator $\mathcal{P}$ up to three–loops can be written as

$$\mathcal{P} = \mathcal{O}_1 - \frac{N^2-6}{2N}(1 + r\,\lambda^2)\,\mathcal{O}_2 \tag{5.15}$$

with

$$r = \frac{\alpha_2}{\alpha_0} = -\frac{9(N^2-9)(q^2-1)^2[(N^4-8N^2-8)(q^4+1)+2(N^4+8)q^2]}{40(N^2-6)[q^4+1+(N^2-2)q^2]^2}\,\zeta(3) \tag{5.16}$$

We note that in the 't Hooft limit, $N \to \infty$ and λ fixed, $\mathcal{O}_2$ dominates and gives the protected operator up to three loops. This is consistent with the fact that, in the absence of mixing, the only primary operators in a given Δ_0 sector are necessarily products of single–trace primaries $\mathrm{Tr}(\Phi_1^m)$ and $\mathrm{Tr}(\Phi_1\Phi_2)$.

The $(4,1,0)$ case: It is interesting to analyze this case in detail since it is the first case where more than one descendant appears.

This sector contains three independent operators

$$\mathcal{O}_1 = \mathrm{Tr}(\Phi_1^4\Phi_2) \quad , \quad \mathcal{O}_2 = \mathrm{Tr}(\Phi_1^3)\mathrm{Tr}(\Phi_1\Phi_2) \quad , \quad \mathcal{O}_3 = \mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_1^2\Phi_2) \tag{5.17}$$

Using the classical equations of motion (2.3), we can write

$$\bar{D}^2\mathrm{Tr}(\Phi_1^3 e^{-gV}\bar{\Phi}_3 e^{gV}) = \mathrm{Tr}\left(\Phi_1^3\frac{\delta W}{\delta\Phi_3}\right) = -ih\,(q-\bar{q})\,[\mathrm{Tr}(\Phi_1^4\Phi_2) - \frac{1}{N}\mathrm{Tr}(\Phi_1^3)\mathrm{Tr}(\Phi_1\Phi_2)] \tag{5.18}$$

$$\bar{D}^2\left[\mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_1 e^{-gV}\bar{\Phi}_3 e^{gV})\right] = \mathrm{Tr}(\Phi_1^2)\mathrm{Tr}\left(\Phi_1\frac{\delta W}{\delta\Phi_3}\right) = -ih\,(q-\bar{q})\,\mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_1^2\Phi_2) \tag{5.19}$$

Therefore, in this case we can consider the two descendants

$$\mathcal{D}_0^{(1)} = \mathcal{O}_1 - \frac{1}{N}\mathcal{O}_2 \qquad , \qquad \mathcal{D}_0^{(2)} = \mathcal{O}_3 \tag{5.20}$$

or any linear combination which realizes an orthogonal basis in the subspace of weight–5 descendants.

As in the previous example it is easy to prove that, given the particular structure (4.1) of the effective superpotential and the way the equations of motion enter the calculation, the linear combinations $\mathcal{D}_0^{(1)}$ and $\mathcal{D}_0^{(2)}$ provide two independent descendants even at the quantum level.

Proceeding as before we consider the linear combination

$$\mathcal{P} = \mathcal{O}_1 + \alpha\,\mathcal{O}_2 + \beta\,\mathcal{O}_3 \tag{5.21}$$

and choose the constants α and β (expanded in powers of λ) by requiring $\mathcal{P}$ to be orthogonal to the two descendants up to two loops.

Solving the constraints $\langle\mathcal{P}_0\bar{\mathcal{D}}_0^{(i)}\rangle_0$ at tree level we determine the correct expression for the operator characterized by a vanishing one–loop anomalous dimension

$$\mathcal{P}_0 = \mathcal{O}_1 - \frac{N^2-12}{3N}\,\mathcal{O}_2 - \frac{2}{N}\,\mathcal{O}_3 \tag{5.22}$$

As in the previous case, this operator is automatically orthogonal to $\mathcal{D}_0^{(1)}$ and $\mathcal{D}_0^{(2)}$ also at one loop and so we expect it to be protected up to two loops.

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The orthogonality at two loops can be imposed exactly as in the previous case and allows to determine the corrections α_2 and β_2. The diagrams contributing are still the ones in figure 3 with one extra free chiral line running between the two vertices. Performing the calculation we find the final expression for the operator protected up to three loops

$$\mathcal{P} = \mathcal{O}_1 - \frac{N^2-12}{3N}\,(1+s_1\,\lambda^2)\,\mathcal{O}_2 - \frac{2}{N}\,(1+s_2\,\lambda^2)\,\mathcal{O}_3 \tag{5.23}$$

where

$$\begin{aligned} s_1 &= \frac{\alpha_2}{\alpha_0} = \frac{(N^2-16)(q^2-1)^2\,[(11N^2+21)(q^4+1)+2(N^2-21)q^2]}{4(N^2-12)[q^4+1+(N^2-2)q^2]^2}\,\zeta(3) \\ s_2 &= \frac{\beta_2}{\beta_0} = -\frac{(N^2-16)(q^2-1)^2[(N^2+5)(q^4+1)+2(N^2-5)q^2]}{8[q^4+1+(N^2-2)q^2]^2}\,\zeta(3) \end{aligned} \tag{5.24}$$

Again, the coefficients depend on N in such a way that in the large N limit only the $\mathcal{O}_2$ operator in (5.17) survives in agreement with the chiral ring content of the theory in the planar limit.

We note that these coefficients, as well as r in (5.16) are real. This is a consequence of the fact that in the sectors studied so far the descendant operators are q–independent and the two–point correlation functions are real.

The previous analysis can be applied to the generic operators of the form $(\Phi_1^J\Phi_2)$. The peculiar pattern $\mathcal{D}_Q \sim \mathcal{D}_0$ for the descendants occurs in any $(J,1,0)$ sector since it only depends on the particular structure of the superpotential and the particular way the equations of motion work for this class of operators. Therefore, the determination of CPO's proceeds as before. In particular, we expect the tree level orthogonality condition to be still sufficient for protection up to two loops since the only one–loop diagram relevant for the calculation would be the vanishing one–loop anomalous dimension diagram in figure 2. At two loops diagrams of the kind drawn in figure 3 should be still the only relevant ones.

Without entering the details of the calculations which would be quite involved and not very illuminating, we can determine the dimension of the corresponding chiral ring subspace, i.e. the number of independent protected operators corresponding to $U(1)$ flavors $(J,1,0)$.

To be definite we consider J even ($J=2p$). In this case the list of chirals we can construct is

$$\begin{array}{lll} \text{single}-\text{trace} & \mathrm{Tr}(\Phi_1^{2p}\Phi_2) & \\ \text{double}-\text{trace} & \mathrm{Tr}(\Phi_1^{m_1})\ \mathrm{Tr}(\Phi_1^{2p-m_1}\Phi_2) & m_1 = 2,\ldots,2p-1 \\ \text{triple}-\text{trace} & \mathrm{Tr}(\Phi_1^{m_1})\ \mathrm{Tr}(\Phi_1^{m_2})\ \mathrm{Tr}(\Phi_1^{2p-m_1-m_2}\Phi_2) & \\ & \qquad m_1 = 2,\ldots,p-1, \quad m_2 = m_1,\ldots,2p-1-m_1 & \\ \quad\vdots & & \\ p-\text{trace} & \mathrm{Tr}(\Phi_1^2)\cdots\mathrm{Tr}(\Phi_1^2)\ \mathrm{Tr}(\Phi_1^2\Phi_2)\ ,\ \mathrm{Tr}(\Phi_1^3)\ \mathrm{Tr}(\Phi_1^2)\cdots\mathrm{Tr}(\Phi_1^2)\ \mathrm{Tr}(\Phi_1\Phi_2) & \end{array} \tag{5.25}$$

In order to find how many independent primaries we can construct out of (5.25) we need first count how many descendants of the form (3.5) we have. As explained in the previous simple examples, given the generic n–trace, $\Delta_0 = J$ sector, null conditions come from considering the operators

$$\mathrm{Tr}(\Phi_1^{m_1}) \cdots \mathrm{Tr}(\Phi_1^{m_{n-1}}) \bar{D}^2 \mathrm{Tr}(\Phi_1^{2p-1-m_1-\cdots-m_{n-1}} e^{-gV} \bar{\Phi}_3 e^{gV}) \tag{5.26}$$

as long as $2p - 1 - m_1 - \cdots - m_{n-1} \geq 1$. In fact, once we act with $\bar{D}^2$ on $\bar{\Phi}_3$ and use the equations of motion (2.3) we generate the linear combination

$$\begin{aligned} &\mathrm{Tr}(\Phi_1^{m_1}) \cdots \mathrm{Tr}(\Phi_1^{m_{n-1}}) \mathrm{Tr}(\Phi_1^{2p-m_1-\cdots-m_{n-1}} \Phi_2) \\ &\qquad - \frac{1}{N} \mathrm{Tr}(\Phi_1^{m_1}) \cdots \mathrm{Tr}(\Phi_1^{m_{n-1}}) \mathrm{Tr}(\Phi_1^{2p-1-m_1-\cdots-m_{n-1}}) \mathrm{Tr}(\Phi_1 \Phi_2) \end{aligned} \tag{5.27}$$

which is then a descendant. Therefore, the complete list of descendants is

$$\begin{array}{ll} \text{single} - \text{trace} & \bar{D}^2\, \mathrm{Tr}(\Phi_1^{2p-1} e^{-gV} \bar{\Phi}_3 e^{gV}) \\ \text{double} - \text{trace} & \bar{D}^2 \left[\mathrm{Tr}(\Phi_1^{m_1})\, \mathrm{Tr}(\Phi_1^{2p-1-m_1} e^{-gV} \bar{\Phi}_3 e^{gV}) \right] \qquad\qquad m_1 = 2, \ldots, 2p-2 \\ \text{triple} - \text{trace} & \bar{D}^2 \left[\mathrm{Tr}(\Phi_1^{m_1})\, \mathrm{Tr}(\Phi_1^{m_2})\, \mathrm{Tr}(\Phi_1^{2p-1-m_1-m_2} e^{-gV} \bar{\Phi}_3 e^{gV}) \right] \\ & \qquad\qquad m_1 = 2, \ldots, p-1, \quad m_2 = m_1, \ldots, 2p-2-m_1 \\ \quad\vdots & \\ p\text{–trace} & \bar{D}^2 \left[\mathrm{Tr}(\Phi_1^2) \cdots\ \mathrm{Tr}(\Phi_1^2)\, \mathrm{Tr}(\Phi_1 e^{-gV} \bar{\Phi}_3 e^{gV}) \right] \end{array} \tag{5.28}$$

Counting how many operators we have in (5.25) and subtracting the number of descendants in (5.28) we find that the number of protected chiral operators is $\sum_{n=2}^{p} X_n$ where X_n is the number of partitions of $(2p-1)$ objects into $(n-1)$ boxes with at least 2 objects per box. Analogously, the number of chiral primary operators for J odd is $\sum_{n=2}^{p+1} X_n$.

This result is consistent with the number of primary operators which survive in the large N limit where mixing effects disappear and the chiral ring reduces to products of single–trace operators $\mathrm{Tr}(\Phi_1^k)$, $\mathrm{Tr}(\Phi_1 \Phi_2)$.

5.2 The $(2, 2, 0)$ flavor

In the class of more general operators with weights $(J_1, J_2, 0)$ we consider the particular case $J_1 = J_2 = 2$. This sector contains four operators, two single– and two double–traces

$$\begin{aligned} \mathcal{O}_1 &= \mathrm{Tr}(\Phi_1^2 \Phi_2^2) \qquad , \qquad \mathcal{O}_2 = \mathrm{Tr}(\Phi_1 \Phi_2 \Phi_1 \Phi_2) \\ \mathcal{O}_3 &= \mathrm{Tr}(\Phi_1^2) \mathrm{Tr}(\Phi_2^2) \qquad , \qquad \mathcal{O}_4 = \mathrm{Tr}(\Phi_1 \Phi_2) \mathrm{Tr}(\Phi_1 \Phi_2) \end{aligned} \tag{5.29}$$

Using the classical equations of motion (2.3), we can write

$$\begin{aligned} \bar{D}^2 \left[\mathrm{Tr}(\Phi_1 \Phi_2 e^{-gV} \bar{\Phi}_3 e^{gV}) - \mathrm{Tr}(\Phi_2 \Phi_1 e^{-gV} \bar{\Phi}_3 e^{gV}) \right] &= -ih(q + \bar{q}) [\mathcal{O}_2 - \mathcal{O}_1] \\ \bar{D}^2 \left[\mathrm{Tr}(\Phi_1 \Phi_2 e^{-gV} \bar{\Phi}_3 e^{gV}) + \mathrm{Tr}(\Phi_2 \Phi_1 e^{-gV} \bar{\Phi}_3 e^{gV}) \right] &= -ih(q - \bar{q}) [\mathcal{O}_1 + \mathcal{O}_2 - \frac{2}{N} \mathcal{O}_4] \end{aligned} \tag{5.30}$$

We note that on the right hand side of these equations the q–dependence is still factored out as it happened in the previous cases (see eqs. (5.2), (5.19)). Therefore, tree level descendants can be defined as linear combinations

$$\begin{aligned}\mathcal{D}_0^{(1)} &= \mathcal{O}_2 - \mathcal{O}_1 \\ \mathcal{D}_0^{(2)} &= \mathcal{O}_1 + \mathcal{O}_2 - \frac{2}{N}\mathcal{O}_4\end{aligned} \tag{5.31}$$

Because of their q–independence these operators correspond indeed to a suitable choice of quantum descendants.

The general structure of a chiral primary operator in this sector is

$$\mathcal{P} = \alpha\mathcal{O}_1 + \beta\,\mathcal{O}_2 + \gamma\,\mathcal{O}_3 + \delta\,\mathcal{O}_4 \tag{5.32}$$

where the coefficients are determined order by order by the orthogonality conditions $\langle\mathcal{P}\;\bar{\mathcal{D}}_0^{(1)}\rangle$ and $\langle\mathcal{P}\bar{\mathcal{D}}_0^{(2)}\rangle$. Having two conditions for four unknowns we expect to single out two protected operators.

At tree level, for the particular choice $\alpha_0 = 2, \beta_0 = 1$ and $\alpha_0 = 1, \beta_0 = -1$, we find

$$\begin{aligned}\mathcal{P}^{(1)} &= 2\mathcal{O}_1 + \mathcal{O}_2 - \frac{N^2-6}{2N}(\mathcal{O}_3 + 2\mathcal{O}_4) \\ \mathcal{P}^{(2)} &= \mathcal{O}_1 - \mathcal{O}_2 - \frac{N}{4}\mathcal{O}_3 + N\mathcal{O}_4\end{aligned} \tag{5.33}$$

These are one–loop protected operators and coincide with the ones found in [21]. They are not orthogonal but a basis can be easily constructed by considering linear combinations.

According to the general pattern already discussed for the previous cases we expect the operators (5.33) to be protected up to two loops. The condition for these operators to be protected up to three loops requires instead nontrivial λ^2–corrections to (5.33) which can be determined by solving the orthogonality constraints at this order. The diagrams contributing nontrivially to the 2–point functions are still the ones in figure 3. Since the final expressions are quite unreadable, we find convenient to fix $\alpha_2 = \beta_2 = 0$ for both the CPO's and we obtain

$$\begin{aligned}\mathcal{P}^{(1)} &= 2\mathcal{O}_1 + \mathcal{O}_2 - \frac{N^2-6}{2N}(1 + t_1\,\lambda^2)\mathcal{O}_3 - \frac{N^2-6}{N}(1 + t_2\,\lambda^2)\mathcal{O}_4 \\ \mathcal{P}^{(2)} &= \mathcal{O}_1 - \mathcal{O}_2 - \frac{N}{4}(1 + u_1\,\lambda^2)\mathcal{O}_3 + N(1 + u_2\,\lambda^2)\mathcal{O}_4\end{aligned} \tag{5.34}$$

where

$$\begin{aligned}t_1 &= -\frac{9(N^2-9)(q^2-1)^2[(N^4-6N^2-4)(q^4+1) + 2(N^4-2N^2+4)q^2]}{20(N^2-6)[q^4+1+(N^2-2)q^2]^2}\,\zeta(3) \\ t_2 &= \frac{9(N^2-9)(N^2+2)(q^2-1)^4}{10(N^2-6)[q^4+1+(N^2-2)q^2]^2}\,\zeta(3)\end{aligned} \tag{5.35}$$

and

$$u_1 = -\frac{9(q^2-1)^2[(N^6-9N^4-16N^2+18)(q^4+1)+2(N^6-14N^4+34N^2-18)q^2]}{20N^2[q^4+1+(N^2-2)q^2]^2}\zeta(3)$$
$$u_2 = \frac{9(q^2-1)^2[(N^4-31N^2-18)(q^4+1)-2(7N^4-13N^2-18)q^2]}{40N^2[q^4+1+(N^2-2)q^2]^2}\zeta(3) \qquad (5.36)$$

6. Chiral Primary Operators in the spin–3 sector

This sector contains operators of the form $(\Phi_1^k\Phi_2^l\Phi_3^m)$ with all possible trace structures.

The simplest case is for $k = l = m = 1$ and involves the two weight–3 operators

$$\mathcal{O}_1 = \mathrm{Tr}(\Phi_1\Phi_2\Phi_3) \qquad , \qquad \mathcal{O}_2 = \mathrm{Tr}(\Phi_1\Phi_3\Phi_2) \qquad (6.1)$$

As already mentioned, the correct one–loop expression for the protected operator has been determined in [19] by computing directly the anomalous dimension at that order. It turns out that the protected operator is a linear combination of the two operators (6.1) with coefficient α as in (3.2). The result has been confirmed in [21] by using a simplified approach based on the evaluation of the difference between the one–loop two–point function of the deformed theory and the one for the $\mathcal{N} = 4$ case. This approach is very convenient since it avoids computing many graphs containing gauge vertices but, as recognized by the authors, in this case it cannot be pushed beyond one loop.

Using our procedure, we can easily re-derive the Freedman–Gursoy result by working at tree level and extend it to two–loops by performing a one–loop calculation. The correct application of our procedure beyond this order would require a substantial modification in the definition of quantum chiral ring (3.9) since in this sector descendants of Konishi–like operators are present and the equations of motion need be supplemented by the Konishi anomaly term. As a consequence the corresponding chiral ring sector necessarily contains operators depending on $W^\alpha W_\alpha$.

In fact, from the anomalous conservation equation for the Konishi current we can write

$$\bar{D}^2\mathrm{Tr}(e^{-gV}\bar{\Phi}_i e^{gV}\Phi_i) = -3ih[q\,\mathrm{Tr}(\Phi_1\Phi_2\Phi_3) - \bar{q}\mathrm{Tr}(\Phi_1\Phi_3\Phi_2)] + \frac{1}{32\pi^2}\mathrm{Tr}(W^\alpha W_\alpha) \qquad (6.2)$$

We remind that in our conventions $W_\alpha = i\bar{D}^2(e^{-gV}D_\alpha e^{gV})$ and it is at least of order g. From the previous identity it follows that a descendant operator has to be constructed out of the two operators (6.1) plus the anomaly term

$$\mathcal{D}_0 = q\mathcal{O}_1 - \bar{q}\mathcal{O}_2 + \frac{i}{96\pi^2\,h}\mathrm{Tr}(W^\alpha W_\alpha) \qquad (6.3)$$

However, since the operator $\mathrm{Tr}(W^\alpha W_\alpha)$ is of order g^2 and has vanishing tree level two–point function with $\mathcal{O}_1$ and $\mathcal{O}_2$ it does not enter the orthogonality conditions at tree level and one–loop. Therefore we can safely use our procedure to find CPO's up to two loops forgetting about the anomaly.

Thus we consider the linear combination

$$\mathcal{P}_0 = \mathcal{O}_1 + \alpha_0\,\mathcal{O}_2 \qquad (6.4)$$

for any value of $\alpha_0 \neq -\bar{q}^2$. In order to determine the exact expression for the CPO at one–loop we need impose the operator to be orthogonal to the descendant (6.3) at tree level. A simple calculation proves that $\langle \mathcal{P}_0 \bar{\mathcal{D}}_0 \rangle_0 = 0$ iff α_0 is given in (3.2), in agreement with the result of [19].

At one loop first we need determine the correct expression for the descendant at this order. As it follows from the calculations of section 3 at one loop the effective superpotential is proportional to the tree level W and the corresponding descendant operator is still proportional to $\mathcal{D}_0$ in eq. (6.3). Given the generic linear combination $\mathcal{P} = \mathcal{O}_1 + (\alpha_0 + \alpha_1 \lambda)\, \mathcal{O}_2$ we then impose the orthogonality condition up to order λ to uniquely determine α_1 as in (5.12). As in the previous examples, if α_0 is given in (3.2) the α_1 coefficient is identically zero being this a consequence of the one–loop protection of $\mathcal{P}_0$. Therefore the expression (6.4) with α_0 given in (3.2) corresponds to the protected chiral operator up to two loops.

The next case we investigate is for $k = 2$, $l = m = 1$. There are five operators

$$\begin{aligned} &\mathcal{O}_1 = \mathrm{Tr}(\Phi_1^2 \Phi_2 \Phi_3) \quad , \quad \mathcal{O}_2 = \mathrm{Tr}(\Phi_1^2 \Phi_3 \Phi_2) \quad , \quad \mathcal{O}_3 = \mathrm{Tr}(\Phi_1 \Phi_2 \Phi_1 \Phi_3) \\ &\mathcal{O}_4 = \mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_2 \Phi_3) \quad , \quad \mathcal{O}_5 = \mathrm{Tr}(\Phi_1 \Phi_2)\mathrm{Tr}(\Phi_1 \Phi_3) \end{aligned} \tag{6.5}$$

Using the classical equations of motion (2.3) we can write three descendants

$$\begin{aligned} \mathcal{D}_0^{(1)} &= q\mathcal{O}_3 - \bar{q}\mathcal{O}_2 - \frac{1}{N}(q - \bar{q})\mathcal{O}_5 \\ \mathcal{D}_0^{(2)} &= q\mathcal{O}_1 - \bar{q}\mathcal{O}_3 - \frac{1}{N}(q - \bar{q})\mathcal{O}_5 \\ \mathcal{D}_0^{(3)} &= q\mathcal{O}_1 - \bar{q}\mathcal{O}_2 - \frac{1}{N}(q - \bar{q})\mathcal{O}_4 \end{aligned} \tag{6.6}$$

We expect to find out two protected operators of the form

$$\mathcal{P} = \alpha\, \mathcal{O}_1 + \beta\, \mathcal{O}_2 + \gamma\, \mathcal{O}_3 + \delta\, \mathcal{O}_4 + \epsilon\, \mathcal{O}_5 \tag{6.7}$$

By imposing the tree-level orthogonality condition with respect to the three $\mathcal{D}_0^{(i)}$ we can fix for instance γ, δ and ϵ in terms of α and β. The calculation proceeds exactly as in the previous case and we find

$$\begin{aligned} \gamma &= \frac{\alpha[q^4 - 2q^2 + 1 - N^2] - \beta[(1 - N^2)q^4 - 2q^2 + 1]}{N^2(q^4 - 1)} \\ \delta &= \frac{\alpha[(N^2 + 2)q^4 + 2(N^2 - 2)q^2 + N^4 - 5N^2 + 2]}{2N^3(q^4 - 1)} \\ &\quad - \frac{\beta[(N^4 - 5N^2 + 2)q^4 + 2(N^2 - 2)q^2 + N^2 + 2]}{2N^3(q^4 - 1)} \\ \epsilon &= \frac{\alpha[2(N^2 + 1)q^4 + (N^4 - 4)q^2 + N^4 - 4N^2 + 2]}{N^3(q^4 - 1)} \\ &\quad - \frac{\beta[(N^4 - 4N^2 + 2)q^4 + (N^4 - 4)q^2 + 2(N^2 + 1)]}{N^3(q^4 - 1)} \end{aligned} \tag{6.8}$$

We expect these operators to have a vanishing anomalous dimension at one loop. If we set $\alpha = \beta = 1$ and $\alpha = -\beta = 1$, we recover the two protected operators found in [21].

As in the previous cases, the operators $\mathcal{D}_0^{(1)}$, $\mathcal{D}_0^{(2)}$ and $\mathcal{D}_0^{(3)}$ keep being good descendants at one loop. Moreover, the one–loop orthogonality conditions do not modify the CPO's (6.7), (6.8) and we expect these operators to have a vanishing two–loop anomalous dimension.

If we were to push our calculation beyond this order we should first determine the descendant operators at two loops. It is easy to realize that in this case the relation $\mathcal{D}_Q \sim \mathcal{D}_0$ does not hold anymore, for two simple reasons:
1) At higher orders the Konishi anomaly cannot be ignored anymore. In particular, the correct expression for the descendant operators from two loops on will have a nontrivial dependence on $(W^\alpha W_\alpha)$.
2) Differently from the spin–2 case, the nontrivial corrections to the effective superpotential which appear at two loops determine nontrivial corrections to the descendants since in this case they depend on q not only through an overall coefficient (see eq. (6.6)).

7. The full Leigh–Strassler deformation

From a field theory point of view it is interesting to investigate the quantum properties of the full Leigh-Strassler $\mathcal{N} = 1$ deformation of the $\mathcal{N} = 4$ SYM theory given by the action [6]

$$\begin{aligned} S = & \int d^8z \mathrm{Tr}(e^{-gV}\bar{\Phi}_i e^{gV}\Phi^i) + \frac{1}{2g^2}\int d^6z \mathrm{Tr}(W^\alpha W_\alpha) \\ & + \left\{ ih \int d^6z \mathrm{Tr}(q\,\Phi_1\Phi_2\Phi_3 - \bar{q}\,\Phi_1\Phi_3\Phi_2) + \frac{ih'}{3}\int d^6z \mathrm{Tr}(\Phi_1^3 + \Phi_2^3 + \Phi_3^3) \ + \ \text{h.c.} \right\} \end{aligned} \tag{7.1}$$

The superpotential now breaks the original $SU(4)$ R–symmetry to $U(1)_R$ and no extra $U(1)$'s are left. However, the action is still invariant under the cyclic permutation of (Φ_1, Φ_2, Φ_3) and the symmetry (2.2). Moreover, a second Z_3 is left corresponding to

$$(\Phi_1, \Phi_2, \Phi_3) \quad \rightarrow \quad (\Phi_1, z\Phi_2, z^2\Phi_3) \tag{7.2}$$

where z is a cubic root of unity.

The equations of motion derived from (7.1) are

$$\begin{aligned} \bar{D}^2(e^{-gV}\bar{\Phi}_1^a e^{gV}) &= -ih\Phi_2^b\Phi_3^c\,[q(abc) - \bar{q}(acb)] - ih'\Phi_1^b\Phi_1^c(abc) \\ \bar{D}^2(e^{-gV}\bar{\Phi}_2^b e^{gV}) &= -ih\Phi_1^a\Phi_3^c\,[q(abc) - \bar{q}(acb)] - ih'\Phi_2^a\Phi_2^c(abc) \\ \bar{D}^2(e^{-gV}\bar{\Phi}_3^c e^{gV}) &= -ih\Phi_1^a\Phi_2^b\,[q(abc) - \bar{q}(acb)] - ih'\Phi_3^a\Phi_3^b(abc) \end{aligned} \tag{7.3}$$

As discussed in [60, 6] the request for the anomalous dimensions of the elementary chiral superfields to vanish guarantees the theory to be superconformal invariant. Since the three chirals have the same anomalous dimension due to the cyclic Z_3 symmetry, superconformal invariance requires a single condition $\gamma(g,h,h',\beta) = 0$ and we find a three–dimensional complex manifold of fixed points.

In general we do not know the superconformal condition exactly. However it is possible to perform a perturbative analysis and define the superconformal theory order by order in the couplings.

To this purpose we evaluate the anomalous dimension of the chiral superfield Φ_i up to two loops. The calculation can be carried on exactly as in the case of $h' = 0$ by taking into account that compared to the previous case the present action contains three extra chiral vertices of the form $\frac{ih'}{6} d_{abc}\Phi_i^a\Phi_i^b\Phi_i^c$, $i = 1, 2, 3$.

As long as we deal with diagrams which do not contain the new h' vertices we have exactly the same contributions as in the $h' = 0$ theory [19, 20]. We only need evaluate all the diagrams which contain these extra vertices.

At one loop, besides the h–chiral and the mixed gauge–chiral self–energy diagrams [20] we have a h'–chiral self–energy graph whose contribution is proportional to $|h'|^2$. This new diagram modifies the one–loop superconformal condition (2.4) as

$$\left[|h|^2\left(1 - \frac{1}{N^2}\,|q - \bar{q}|^2\right) + |h'|^2\frac{N^2-4}{2N^2}\right] = g^2 \tag{7.4}$$

in agreement with [61, 7, 14]. As for the $h' = 0$ case it is easy to verify that the one–loop condition is sufficient to guarantee the vanishing of the beta functions (i.e. superconformal invariance) up to two loops.

Once the theory is made finite we are interested in the perturbative evaluation of *finite* corrections to the superpotential. In this case the symmetries of the theory force the effective superpotential to have the form

$$W_{eff} = ih\int d^6z\,\mathrm{Tr}[b(q)\,\Phi_1\Phi_2\Phi_3 + b(-\bar{q})\,\Phi_1\Phi_3\Phi_2] + \frac{ih'}{3}d\int d^6z\,\mathrm{Tr}(\Phi_1^3+\Phi_2^3+\Phi_3^3) + \text{h.c.} \tag{7.5}$$

where the coefficients b and d are determined as double power expansions in the couplings h and h' [7]. In particular, the invariance under cyclic permutations of the superfields requires the d correction to be the same for the three Φ_i^3 terms, whereas the other global symmetries force the particular q dependence of the corrections to $(\Phi_1\Phi_2\Phi_3)$ and $(\Phi_1\Phi_3\Phi_2)$. We note that in this case we cannot apply the previous arguments (see the discussion after eq. (4.1)) to state that $b(-\bar{q}) = -\overline{b(q)}$ since the perturbative corrections to $(\Phi_1\Phi_2\Phi_3)$ and $(\Phi_1\Phi_3\Phi_2)$ are not always proportional to q times functions of q^2. In fact, it is still true that diagrams contributing to the effective potential contain an even number of extra chiral vertices compared to the tree level diagrams, but part of these vertices could be h'–vertices not carrying any q–dependence.

The topologies of diagrams contributing to the superpotential up to two loops are still the ones in figure 1 where now chiral vertices may be either h or h' vertices. Performing the explicit calculation as in section 4 we discover that at one loop the various terms in the superpotential do not mix and receive separate corrections still proportional to the classical

[7]Here we use the superconformal condition (7.4) to express g^2 as a function of h and h'. Any other choice would be equally acceptable.

terms. Precisely, we find that $W_{eff}^{(1)}$ coincides with W, up to an overall constant coefficient. This is also true at two loops for the diagrams 1c), 1d) and 1e), whereas the diagram 1g) with all possible configurations of h and h' vertices mixes nontrivially the various terms of the superpotential. Similarly to what happens for the β–deformed theory, this leads to a nontrivial correction $W_{eff}^{(2)}$ which has the form (7.5) but with the b and d coefficients nontrivially corrected by functions of q and N. We then expect descendant operators to get modified at this order as in the previous case (see discussion around eq. (4.6)).

The exact supergravity dual of the theory (7.1) is still unknown even if few steps towards it have been undertaken in [4]. However, it is interesting to investigate the nature of composite operators of the superconformal field theory waiting for the discovery of the exact correspondence of these operators to superstring states.

The chiral ring for the h'-deformed theory is not known in general (however, see [11]). Compared to the chiral ring of the β–deformed theory ($h' = 0$) which contains operators of the form $\mathrm{Tr}(\Phi_i^J)$, $\mathrm{Tr}(\Phi_1^J \Phi_2^J \Phi_3^J)$ plus the particular operators $\mathrm{Tr}(\Phi_i \Phi_j)$, $i \neq j$, we expect the chiral ring of the present theory to be more complicated because of the lower number of global symmetries present.

Here we exploit the general procedure described in section 3 to move the first steps towards the determination of chiral primary operators. In particular, we concentrate on the first simple cases of matter chiral operators with dimensions $\Delta_0 = 2, 3$ and study how turning on the h'-interaction may affect their quantum properties. We then take advantage of these results to make a preliminary discussion of the CPO content for generic scale dimensions.

7.1 Chiral ring: The $\Delta_0 = 2$ sector

Weight–2 chiral operators are $\mathrm{Tr}(\Phi_i^2)$ and $\mathrm{Tr}(\Phi_i \Phi_j)$, $i \neq j$. These operators can be classified as in table 1 according to their charge $\mathcal{Q}$ with respect to the Z_3 symmetry (7.2).

$\mathcal{Q} = 0$	$\mathcal{Q} = 1$	$\mathcal{Q} = 2$
$\mathcal{O}_{11} = \mathrm{Tr}(\Phi_1^2)$	$\mathcal{O}_{33} = \mathrm{Tr}(\Phi_3^2)$	$\mathcal{O}_{22} = \mathrm{Tr}(\Phi_2^2)$
$\mathcal{O}_{23} = \mathrm{Tr}(\Phi_2 \Phi_3)$	$\mathcal{O}_{12} = \mathrm{Tr}(\Phi_1 \Phi_2)$	$\mathcal{O}_{13} = \mathrm{Tr}(\Phi_1 \Phi_3)$

Table 1: Operators with $\Delta_0 = 2$.

The charged sectors can be obtained from the $\mathcal{Q} = 0$ one by successive applications of cyclic Z_3–permutations $\Phi_i \to \Phi_{i+1}$. This is the reason why the three sectors contain the same number of operators. In the $h' = 0$ theory their anomalous dimensions have been computed up to two loops and found to be vanishing [19, 20]. According to our discussion in section 3 this was an expected result since for these operators there is no way to use the equations of motion (2.3) to write them as $\bar{D}^2 X$. Therefore they must be necessarily primaries and belong to the classical chiral ring. Since this sector does not contain descendants this property is mantained at the quantum level. In the $h' = 0$ case these operators have different $U(1)$ flavor charges and do not mix. The matrix of their two–point functions is then diagonal and receives finite corrections at two loops [20].

The same analysis can be applied in the present case. Again, there is no way to write these operators as descendants by using the classical equations of motion (7.3). Therefore, we expect them to belong to the chiral ring.

In order to check that these operators do not get renormalized but their correlators might receive finite corrections we compute directly their two–point functions.

The smaller number of global symmetries surviving the h'–deformation do not prevent the operators to mix. For instance the operator $\mathrm{Tr}(\Phi_1^2)$ can mix with $\mathrm{Tr}(\Phi_2\Phi_3)$ since they have the same charge under the Z_3 symmetry (7.2). Therefore, we need compute the non–diagonal matrix of their two–point functions.

To this purpose we concentrate on the operators $\mathcal{O}_{11}$ and $\mathcal{O}_{23}$ and evaluate all the correlators up to two loops. The calculation goes exactly as in the $h' = 0$ theory with the understanding of adding contributions from diagrams containing the new h'–vertices.

At one–loop, as in the undeformed [52, 53] and the β–deformed cases [20] we do not find any divergent nor finite contributions to the two–point functions as long as the superconformal condition (7.4) holds.

At two loops the topologies of diagrams which contribute to $\langle \mathcal{O}_{11}\bar{\mathcal{O}}_{11}\rangle$ and $\langle \mathcal{O}_{23}\bar{\mathcal{O}}_{23}\rangle$ are the ones in figure 4.

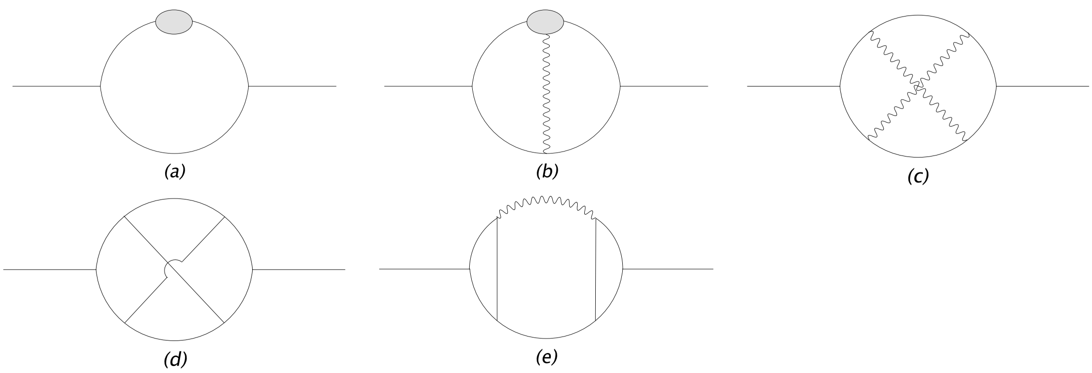

Figure 4: Two–loop diagrams for $\langle \mathcal{O}_{11}\bar{\mathcal{O}}_{11}\rangle$ and $\langle \mathcal{O}_{23}\bar{\mathcal{O}}_{23}\rangle$.

Here the grey bullets indicate two–loop corrections to the chiral propagator and one–loop corrections to the mixed gauge–chiral vertex. Using the superconformal condition (7.4) their q, h, h' dependence disappears and these corrections coincide with the ones of the $\mathcal{N} = 4$ theory [50, 52, 53]. Therefore the first three diagrams give the same kind of contribution to both correlators.

The last two diagrams contain chiral vertices and they instead differ in the two cases for the number of h vs. h' insertions: Diagram 4d) gives contributions proportional to $|h|^4$ and $|h'|^4$ to $\langle \mathcal{O}_{11}\bar{\mathcal{O}}_{11}\rangle$, and contributions proportional to $|h|^4$ and $|h|^2|h'|^2$ to $\langle \mathcal{O}_{23}\bar{\mathcal{O}}_{23}\rangle$. Analogously, diagram 4e) contributes to $\langle \mathcal{O}_{11}\bar{\mathcal{O}}_{11}\rangle$ with a term proportional to $g^2|h'|^2$ and to $\langle \mathcal{O}_{23}\bar{\mathcal{O}}_{23}\rangle$ with $g^2|h|^2$.

Diagrams contributing to the mixed two–point function $\langle \mathcal{O}_{11}\bar{\mathcal{O}}_{23}\rangle$ at two loops are of the type 4d) with two h and two h' vertices (contributions proportional to $\bar{h}^2 h'^2$), with three h and one h' (contributions proportional to $|h|^2\bar{h}'h$) and 4e) with one h and one h' vertices (contributions proportional to $g^2 h\bar{h}'$).

Performing the D–algebra and computing the corresponding loop integrals in momentum space and dimensional regularization, it is easy to verify that the diagrams 4a)–d) have at most $1/\epsilon$ poles which correspond to finite corrections to the two–point functions when transformed back to the configuration space.

The only potential source of anomalous dimension terms would be the graph 4e) since, after D–algebra, the corresponding integral has a $1/\epsilon^2$ pole, that is a $\log(\mu^2 x^2)$ divergence in configuration space. However, when computing the correlators $\langle \mathcal{O}_{11}\bar{\mathcal{O}}_{11}\rangle$ and $\langle \mathcal{O}_{11}\bar{\mathcal{O}}_{23}\rangle$ this diagram gives a vanishing color factor, whereas for the third correlator there is a complete cancellation between the contribution corresponding to a particular configuration of the $\bar{\Phi}_2$, $\bar{\Phi}_3$ lines coming out from the $\bar{\mathcal{O}}_{23}$ vertex and the one with the two lines interchanged (the same happens in the $h' = 0$ theory [20]).

Therefore, all the correlators in configuration space are two–loop finite, the anomalous dimension matrix vanishes and the two operators are protected up to this order.

It is interesting to give the explicit result for the two–loop corrections to the correlators. We find

$$\begin{aligned}\langle \mathrm{Tr}(\Phi_1^2)(z_1)\mathrm{Tr}(\bar{\Phi}_1^2)(z_2)\rangle_{2-\text{loops}} &\sim \frac{\delta^{(4)}(\theta_1-\theta_2)}{[(x_1-x_2)^2]^2}\,\mathcal{F}_1 \\ \langle \mathrm{Tr}(\Phi_2\Phi_3)(z_1)\mathrm{Tr}(\bar{\Phi}_2\bar{\Phi}_3)(z_2)\rangle_{2-\text{loops}} &\sim \frac{\delta^{(4)}(\theta_1-\theta_2)}{[(x_1-x_2)^2]^2}\,\mathcal{F}_2\end{aligned} \tag{7.6}$$

where

$$\begin{aligned}\mathcal{F}_1 = &\left[|h|^4\frac{N^2-4}{N^2}\,|q-\bar{q}|^2\left(\frac{N^2-1}{4N^2}\,|q-\bar{q}|^2-1\right)\right. \\ &\left. + \ |h'|^4\frac{(N^2-20)(N^2-4)}{8N^4} - |h|^2|h'|^2\frac{N^2-4}{2N^2}\left(1-\frac{1}{N^2}\,|q-\bar{q}|^2\right)\right]\end{aligned} \tag{7.7}$$

and

$$\begin{aligned}\mathcal{F}_2 = &\left[|h|^4\frac{N^2-4}{4N^4}\,|q-\bar{q}|^4 + |h'|^4\frac{(N^2-4)^2}{8N^4}\right. \\ &\left. + \ |h|^2|h'|^2\frac{N^2-4}{2N^2}\left(3-\frac{N^2-5}{N^2}\,|q-\bar{q}|^2\right)\right]\end{aligned} \tag{7.8}$$

We note that all the g^4 contributions cancel and we are left with expressions which vanish in the $\mathcal{N}=4$ limit ($\beta = h' = 0$, $|h|^2 = g^2$). Moreover, both the contributions survive in the large N limit in contradistinction to the $h' = 0$ case where $\mathcal{F}_2$ is subleading [20].

7.2 Chiral ring: The $\Delta_0 = 3$ sector

The next sector we investigate contains operators with naive scale dimension $\Delta_0 = 3$. We classify them according to their Z_3–charge as in table 2.

We note that the neutral sector does not contain the same number of operators as the charged ones. This is due to the fact that, in contradistinction to the previous case,

$\mathcal{Q}=0$	$\mathcal{Q}=1$	$\mathcal{Q}=2$
$\mathcal{O}_1 = \mathrm{Tr}(\Phi_1^3)$	$\mathcal{O}_6 = \mathrm{Tr}(\Phi_1^2\Phi_2)$	$\mathcal{O}_9 = \mathrm{Tr}(\Phi_1^2\Phi_3)$
$\mathcal{O}_2 = \mathrm{Tr}(\Phi_2^3)$	$\mathcal{O}_7 = \mathrm{Tr}(\Phi_2^2\Phi_3)$	$\mathcal{O}_{10} = \mathrm{Tr}(\Phi_3^2\Phi_2)$
$\mathcal{O}_3 = \mathrm{Tr}(\Phi_3^3)$	$\mathcal{O}_8 = \mathrm{Tr}(\Phi_3^2\Phi_1)$	$\mathcal{O}_{11} = \mathrm{Tr}(\Phi_2^2\Phi_1)$
$\mathcal{O}_4 = \mathrm{Tr}(\Phi_1\Phi_2\Phi_3)$		
$\mathcal{O}_5 = \mathrm{Tr}(\Phi_1\Phi_3\Phi_2)$		

Table 2: Operators with $\Delta_0 = 3$.

the $\mathcal{Q} = 0$ sector is closed under the application of cyclic permutations $\Phi_i \to \Phi_{i+1}$ and tranformations (2.2). Therefore, we cannot generate the charged sectors from the neutral one by using these mappings.

The charged sectors are also closed under permutations but they get exchanged under transformations (2.2). This is the reason why they still have the same number of operators.

We first focus on the set of operators with $\mathcal{Q} = 0$. As for the $h' = 0$ theory, in this sector the Konishi anomaly enters the game when we try to use the equations of motion to write descendants which involve $\mathcal{O}_4$ and $\mathcal{O}_5$. However, as discussed in section 6, the Konishi anomaly can be neglected as long as we are interested in the construction of CPO's up to two loops. We will then restrict our analysis at this order.

Using the equations of motion (7.3) we can write three descendant operators

$$\begin{aligned} \mathcal{D}^{(1)} &= h\,(q\,\mathcal{O}_4 - \bar{q}\,\mathcal{O}_5) + h'\,\mathcal{O}_1 \\ \mathcal{D}^{(2)} &= h\,(q\,\mathcal{O}_4 - \bar{q}\,\mathcal{O}_5) + h'\,\mathcal{O}_2 \\ \mathcal{D}^{(3)} &= h\,(q\,\mathcal{O}_4 - \bar{q}\,\mathcal{O}_5) + h'\,\mathcal{O}_3 \end{aligned} \tag{7.9}$$

According to the discussion of section 3 we expect to single out two protected operators. We consider the most general linear combination

$$\mathcal{P} = \alpha\,\mathcal{O}_1 + \beta\,\mathcal{O}_2 + \gamma\,\mathcal{O}_3 + \delta\,\mathcal{O}_4 + \epsilon\,\mathcal{O}_5 \tag{7.10}$$

and require tree-level orthogonality to the three descendants. These constraints provide the condition $\alpha = \beta = \gamma \equiv a$ (as expected because of the Z_3 symmetries of this sector) and the extra relation

$$3a\bar{h}'(N^2-4)q + \bar{h}\left[\delta\,\left(N^2-2+2q^2\right) - \epsilon\,\left((N^2-2)q^2+2\right)\right] = 0 \tag{7.11}$$

which can be used to express a in terms of two arbitrary constants.

Any CPO in this sector has then the following form

$$\mathcal{P} = a\,(\mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3) + \delta\,\mathcal{O}_4 + \epsilon\,\mathcal{O}_5 \tag{7.12}$$

An explicit check on its two–point function at one loop leads to $\langle \mathcal{P}\,\bar{\mathcal{P}}\rangle_1$ finite, independently of the choice of δ and ϵ. One can choose the two constants in order to select two mutually orthogonal operators.

As it happened in the previous cases, these operators are guaranteed to be protected up to two loops as a consequence of their one–loop protection plus the result $W_{eff}^{(1)} \sim W$ which insures that the classical descendants (7.9) keep being good descendants also at one loop.

The sectors characterized by Z_3 charges $\mathcal{Q} = 1, 2$ do not contain protected operators. In fact, one can see that any charged operator in table 2 can be written as $\mathcal{O}_i = \bar{D}^2 X_i$ by using the classical equations of motion. We expect this result to be valid at any order of perturbation theory since the structure of the effective superpotential for what concerns its superfield dependence cannot change.

To summarize, in the $\Delta_0 = 3$ sector we have found two protected operators which are linear combinations of $\mathrm{Tr}(\Phi_i^3)$, $i = 1, 2, 3$, $\mathrm{Tr}(\Phi_1\Phi_2\Phi_3)$ and $\mathrm{Tr}(\Phi_1\Phi_3\Phi_2)$. We note that among all possible weight–3 operators these are the only ones which belong to the chiral ring of the β–deformed theory. The rest of weight–3 operators which were descendants for $h' = 0$ keep being descendants.

The protected operators we have found are neutral under the Z_3 symmetry (7.2). As discussed in [11], the neutral sector of the chiral ring (the untwisted sector) coincides with the center of the quantum algebra generated by the F–terms constraints. In particular, for the h'–deformation one element of the center has been constructed explicitly (eq. (4.83) in [11]). This element coincides with one of the CPO's (7.12) we have found, once we set $\mathcal{D}^{(i)} = 0$ in the chiral ring (see eq. (7.9)), use these identities to express the operator $\mathcal{O}_5$ in terms of the other ones and make a suitable choice for the coefficients δ and ϵ.

7.3 Comments on the general structure of the chiral ring

The $\Delta_0 = 2, 3$ sectors studied in the previous section are very peculiar and do not provide enough informations to guess the structure of the sectors for generic scale dimension. In fact, for $\Delta_0 = 2$ no descendants are present and we cannot even apply the orthogonality procedure to construct CPO's. The $\Delta_0 = 3$ sector contains only protected operators which are Z_3 neutral and are linear combinations of "old" CPO's, that is operators which were protected for $h' = 0$.

A naive generalization of our results to higher dimensional sectors would lead to the conjecture that the chiral ring for the h'–deformed theory, at least for what concerns its neutral sector with $\Delta_0 = 3J$, would be given by linear combinations of $\mathrm{Tr}(\Phi_i^{3J})$ and $\mathrm{Tr}(\Phi_1^J\Phi_2^J\Phi_3^J)$. However, we expect more general operators of the form $\mathrm{Tr}(\Phi_1^{3J-m-n}\Phi_2^m\Phi_3^n)$, $m + 2n = \mathrm{mod}(3)$ to appear. Moreover, nontrivial Z_3–charged sectors should appear for $\Delta_0 = 3J$ even if they are absent in the particular case $\Delta_0 = 3$.

To investigate these issues we should extend our analysis to higher dimensional sectors and this would require quite a bit of technical effort. However, without entering any calculative detail, but simply performing dimensional and Z_3–charge balances we can figure out few general properties of the $\mathcal{Q}$–sectors of the chiral ring.

We consider the generic chiral operator $\mathcal{O}_1 = (\Phi_1^a\Phi_2^b\Phi_3^c)$ for any trace structure with scale dimension $\Delta_0 = a+b+c$ and Z_3–charge $\mathcal{Q}_1 \equiv b+2c$ with respect to the symmetry (7.2).

We now perform $\Phi_i \leftrightarrow \Phi_j$ exchanges according to the symmetry (2.2) and Z_3 permutations. In this way of doing we generate all the operators with the same trace structure in a given Δ_0 sector. Let us consider for example the operators $\mathcal{O}_2 = (\Phi_2^a \Phi_1^b \Phi_3^c)$ and $\mathcal{O}_3 = (\Phi_3^a \Phi_1^b \Phi_2^c)$ obtained by a $\Phi_1 \leftrightarrow \Phi_2$ exchange and a cyclic permutation, respectively. They have charges $\mathcal{Q}_2 = a + 2c$ and $\mathcal{Q}_3 = 2a + c$. It is easy to see that if $\Delta_0 = 3J$ then $\mathcal{Q}_2 = \mathcal{Q}_3 = 0 \ (\mathrm{mod}(3))$ *iff* $\mathcal{Q}_1 = 0 \ (\mathrm{mod}(3))$. This property holds for any operator that we can construct from $\mathcal{O}_1$ by the application of the two discrete symmetries. On the other hand, if $\mathcal{Q}_1 = 1, 2(\mathrm{mod}(3))$ operators obtained from it by cyclic permutations still mantain the same charge, but the application of field exchanges (2.2) map charge–1 operators into charge–2 operators and viceversa.

Therefore, for $\Delta_0 = 3J$ the $\mathcal{Q} = 0$ class is closed under the action of Z_3–permutations and (2.2) symmetry, and being independent, may contain a different number of operators compared to the charged sectors which instead are related by (2.2) mappings. In particular, as it happens for $\Delta_0 = 3$ charged classes of the chiral ring might be empty while the corresponding neutral one is not.

If $\Delta_0 \neq 3J$ a simple calculation leads to the conclusion that starting from operators with zero Z_3–charge we generate operators with $\mathcal{Q} = 1$ by applying $\Phi_1 \leftrightarrow \Phi_2$ if $\Delta_0 = 3J+1$ and a cyclic permutation if $\Delta_0 = 3J+2$. Correspondingly, we obtain operators with $\mathcal{Q} = 2$ by applying a cyclic permutation in the first case and a $\Phi_1 \leftrightarrow \Phi_2$ exchange in the second case. Therefore, in any sector with $\Delta_0 \neq 3J$ the number of operators with $\mathcal{Q} = 1$ is the same as the ones with $\mathcal{Q} = 2$ and coincides with the number of neutral operators.

If we apply the same reasoning to the descendant operators of each sector (to simplify the analysis we work at large N to avoid mixing among different trace structures) we discover that every time $\Delta_0 \neq 3J$ the descendants of the charged classes can be obtained from the neutral ones by field exchanges. As a consequence, the three classes contain the same number of descendants and then the *same* number of protected operators.

To summarize, the sectors of the chiral ring behave differently according to their scale dimension: If $\Delta_0 \neq 3J$ the corresponding operators are equally split into the three $\mathcal{Q}$ classes. On the contrary, if $\Delta_0 = 3J$ the neutral class is independent and may contain a different number of CPO's.

As a further example we have studied the $\Delta_0 = 4$ operators. In the large N limit and at the lowest order in perturbation theory we have found that the neutral single–trace sector contains one independent CPO (we have eight single–trace chirals and seven descendants). Therefore, we conclude that also the charged sectors contain one single protected operator and we know how to construct it once we have found the $\mathcal{Q} = 0$ operator explicitly. In the single–trace sector the protected operator turns out to be a linear combination of

$$\begin{array}{l}
\mathrm{Tr}(\Phi_1^4) \\
\mathrm{Tr}(\Phi_1 \Phi_2^3) \quad , \quad \mathrm{Tr}(\Phi_1 \Phi_3^3) \quad , \quad \mathrm{Tr}(\Phi_2^2 \Phi_3^2) \quad , \quad \mathrm{Tr}(\Phi_2 \Phi_3 \Phi_2 \Phi_3) \\
\mathrm{Tr}(\Phi_1^2 \Phi_2 \Phi_3) \quad , \quad \mathrm{Tr}(\Phi_1^2 \Phi_3 \Phi_2) \quad , \quad \mathrm{Tr}(\Phi_1 \Phi_2 \Phi_1 \Phi_3)
\end{array} \tag{7.13}$$

It remains the open question whether for $\Delta_0 = 3J$, $J > 1$, the charged sectors are trivial as in the weight–3 case. A systematic analysis of the charged protected operators is a difficult task in general. However, working at large N it is easy to realize that for J *even* and $J > 1$, there are nontrivial protected operators for $\mathcal{Q} = 1$ and $\mathcal{Q} = 2$. These are operators with the $3J$ chiral superfields split into the maximal number of traces allowed by $SU(N)$, i.e. $3J/2$. In fact, for these operators it is impossible to exploit the equations of motion and write them as descendants. For J *odd* the same arguments do not lead to any definite conclusion. However, we expect to generate nontrivial charged protected operators by multiplying the neutral CPO's of weight 3 previously constructed by $3(J-1)/2$ traces containing two operators each and carrying the right Z_3 charge.

8. Conclusions

In this paper we have considered $\mathcal{N}=1$ $SU(N)$ SYM theories obtained as marginal deformations of the $\mathcal{N}=4$ theory. In particular, we have focused on the perturbative structure of the matter (not gauge) quantum chiral ring defined as in (3.9) in terms of the effective superpotential. According to our general prescription, CPO's can be determined by imposing order by order the orthogonality condition (3.6) to all the descendants of a given sector. This requires constructing first the descendants as a power expansion in the couplings. According to the definition (3.9), this can be easily accomplished once the effective superpotential is known at a given order.

For the Lunin–Maldacena β–deformed theory (2.1) we have studied quite extensively the spin–2 sector of the theory. For the particular examples of weights $(J,1,0)$ and $(2,2,0)$ we have considered, a special pattern arises which allows for a drastic simplification in the study of the orthogonality condition: In any of these sectors descendants can be always constructed at tree level which turn out to be good independent descendants even at the quantum level. This is due to the particular form (4.1) of the superpotential and the peculiar way the equations of motion work which allow for constructing q–independent descendants, insensible to the quantum corrections of the theory. This property persists even for other examples of the form $(J_1,J_2,0)$. Therefore, we conjecture that it might be a property of the entire spin–2 sector: For any weight $(J_1,J_2,0)$ quantum descendant operators can be constructed which coincide with the descendants determined classically.

We have then studied the spin–3 sector. In this case the determination of quantum descendants of weights (J_1,J_2,J_3) cannot ignore the Konishi anomaly term. Being its effect of order λ it only enters nontrivially the orthogonality condition from two loops on, that is it will affect the form of the protected operators at least at three loops. For weights $(1,1,1)$ and $(2,1,1)$ we have determined the CPO's up to two loops. In particular, for the first case we have proved that up to this order the correct CPO is the one found in [19]. Higher order calculations would require computing two–point correlation functions between matter chiral operators and $\mathrm{Tr}(W^\alpha W_\alpha)$. It would be interesting to pursue this direction since it represents the first case where the descendant operators, apart from acquiring an explicit dependence on the Konishi anomaly term, get modified nontrivially at the quantum level due to the nontrivial corrections to the superpotential which start appearing at order λ^2.

We have extended our procedure to the study of protected operators for the full Leigh–Strassler deformation. We can think of this theory as a marginal perturbation of the β–deformed theory induced by the h'–terms in (7.1). In this case the determination of the complete chiral ring is a difficult task and only few insights have been discussed in [11].

We have moved few steps in this direction by studying perturbatively the simple $\Delta_0 = 2, 3$ sectors. For operators of scale dimension two we have found that the h'–deformed theory has still the same CPO's as the $h' = 0$ one, i.e. $\mathrm{Tr}(\Phi_i^2)$ and $\mathrm{Tr}(\Phi_i \Phi_j)$, $i \neq j$.

For the $\Delta_0 = 3$ sector we have found a two–dimensional plane of CPO's given as linear combinations of the CPO's of the corresponding $h' = 0$ theory, i.e. $\mathrm{Tr}(\Phi_i^3)$ and $\mathrm{Tr}(\Phi_1\Phi_2\Phi_3)$. In fact, in this case the lower number of global symmetries surviving the deformation allows for mixing among the operators who were protected in the previous case and belonged to different $U(1) \times U(1)$ sectors. The class of protected operators we have found contains the central element of the quantum algebra proposed in [11].

What turns out is that in the $\Delta_0 = 2$ sector the chiral ring is made by operators which are both charged and neutral with respect to the Z_3–symmetry (7.2) that the theory inherits from the parent $h' = 0$ theory. On the other hand, in the $\Delta_0 = 3$ sector *all* CPO's we can construct are neutral under (7.2). The generalization of our results to higher dimensional sectors leads to the result that the chiral ring for the h'–deformed theory can be divided into two subsets: Sectors with scale dimension $\Delta_0 = 3J$ have an independent $\mathcal{Q} = 0$ class which may contain in general a different number of CPO's. Instead, whenever $\Delta_0 \neq 3J$ we can generate the chiral primary operators of the charged classes from neutral CPO's by the use of the other discrete symmetries, i.e. cyclic permutations of the three superfields and the symmetry (2.2). It then follows that the three classes contain the same number of protected operators. In particular, for any non–empty neutral sector (for instance $\Delta_0 = 2, 4$) the corresponding charged ones are nontrivial. Neutral CPO's will be in general linear combinations of operators of the form $\mathrm{Tr}(\Phi_1^{J-m-n}\Phi_2^m\Phi_3^n)$ with $m + 2n = 3p$.

The Z_3 periodicity we have found in the chiral ring structure should have a counterpart in the spectrum of BPS states of the dual supergravity theory. Therefore, it might be of some help in the construction of the dual spectrum.

For *all* the cases we have investigated the CPO's do not get corrected at one–loop, whereas they start being modified at order λ^2. This one–loop non–renormalization found for a large class of chiral operators is probably universal for all the CPO's and might be traced back to the one–loop non–renormalization properties of the theories. Precisely, the conditions (2.4), (7.4) which insure superconformal invariance at one–loop are maintained at two loops, i.e. the superconformal theories at one and two loops are the same. It is then natural to speculate that the corresponding chiral rings should be the same. The theory instead changes at three loops where the superconformal condition gets modified by terms of order λ^2 [21]. Therefore we expect that at this order the chiral ring will be modified by effects of the same order.

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A. Integrals in momentum space

In this appendix we list the results for loop integrals that we have used along the calculations. Working in momentum space and dimensional regularization ($n = 4 - 2\epsilon$) we give the results as ϵ expansions.

We begin by considering the momentum integrals associated to the one–loop and two–loop diagrams in figure 1 for the perturbative corrections to the superpotential.

At one loop, after performing D–algebra, the diagram 1b) gives the standard triangle contribution [62]. Assigning external momenta p_i ($p_1 + p_2 + p_3 = 0$) we have

$$p_3^2 \int \frac{d^n q}{(2\pi)^n} \frac{1}{q^2 (q-p_2)^2 (q+p_1)^2} = \frac{1}{(4\pi)^2} \, \Phi^{(1)} \, (x,y) + \mathcal{O}(\epsilon) \tag{A.1}$$

where

$$x \equiv \frac{p_1^2}{p_3^2} \quad \text{and} \quad y \equiv \frac{p_2^2}{p_3^2} \tag{A.2}$$

The p_3^2 in front of the integral is produced by D–algebra. The function $\Phi^{(1)}(x,y)$ can be represented as a parametric integral

$$\Phi^{(1)}(x,y) = -\int_0^1 \frac{d\xi}{y\,\xi^2 + (1-x-y)\xi + x} \left(\log \frac{y}{x} + 2 \log \xi \right) \tag{A.3}$$

Since we look for a local contribution to the superpotential we are interested in the result of the integral for external momenta set to zero. A consistent way [63] to take the limit of vanishing external momenta is to set $p_i^2 = m^2$ for any i so having $x, y = 1$ and let the IR cut–off m^2 going to zero at the end of the calculation. In the limit we obtain a finite local result [63]

$$-\int_0^1 d\xi \frac{\log \xi(1-\xi)}{1 - \xi(1-\xi)} \tag{A.4}$$

At two loops two types of integrals appear. From diagrams 1c) and 1d) we have integrals of the form

$$\begin{aligned} (p_3^2)^2 \int \frac{d^n q \, d^n r}{(2\pi)^{2n}} & \frac{1}{(r+p_1)^2 (q+p_1)^2 (r-p_2)^2 (q-p_2)^2 r^2 (q-r)^2} = \\ = \frac{1}{(4\pi)^4} \, \Phi^{(2)} \, (x,y) + \mathcal{O}(\epsilon) & \end{aligned} \tag{A.5}$$

with x and y as in (A.2). The function $\Phi^{(2)}(x,y)$ is defined by [62]

$$\Phi^{(2)}(x,y) = -\frac{1}{2} \int_0^1 \frac{d\xi}{y\,\xi^2 + (1-x-y)\xi + x} \log \xi \left(\log \frac{y}{x} + \log \xi \right) \left(\log \frac{y}{x} + 2 \log \xi \right) \tag{A.6}$$

As in the one–loop case, the limit $x, y \to 1$ gives a finite local contribution to the effective superpotential.

From diagrams 1c)–g) this kind of integral also appears

$$p_3^2 \int \frac{d^n q \, d^n r}{(2\pi)^{2n}} \frac{1}{q^2 \, r^2 (q-r)^2 (q-p_3)^2 (r-p_3)^2} = \frac{1}{(4\pi)^4} \, 6\zeta(3) + \mathcal{O}(\epsilon) \tag{A.7}$$

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where one of the external momenta has been already set to zero (in this case we can safely set one of the external momenta to zero from the very beginning since we do not introduce fake IR divergences). This is already the local finite contribution we obtain by setting also $p_3^2 = 0$.

When we deal with two-point correlation functions, at tree-level we have ($k = \Delta_0$ is the free scale dimension of the operators involved and p is the external momentum)

$$\int \frac{d^n q_1 \, \dots \, d^n q_{k-1}}{(2\pi)^{n(k-1)}} \frac{1}{q_1^2 (q_2 - q_1)^2 (q_3 - q_2)^2 \dots (p - q_{k-1})^2}$$
$$= \frac{1}{\epsilon} \left[\frac{1}{(4\pi)^2} \right]^{k-1} \frac{(-1)^k}{[(k-1)!]^2} (p^2)^{k-2-(k-1)\epsilon} + \mathcal{O}(1) \tag{A.8}$$

At two loops we are interested in the four diagrams listed in figure 3. From the graph 3a) we obtain

$$\int \frac{d^n q_3 \, \dots \, d^n q_{k-1}}{(2\pi)^{n(k+1)}} \frac{1}{(q_4 - q_3)^2 \dots (p - q_{k-1})^2} \times$$
$$\int \frac{d^n k \, d^n l \, d^n r \, d^n s \quad r^2 (q_3 - l)^2}{k^2 l^2 (k-l)^2 (r-k)^2 (r-l)^2 (s-l)^2 (r-s)^2 (q_3 - r)^2 (q_3 - s)^2} \tag{A.9}$$
$$= \frac{1}{\epsilon} \left[\frac{1}{(4\pi)^2} \right]^{k+1} \frac{(-1)^k (k-1)}{[(k-1)!]^2 (k+1)} [6\zeta(3) - 20\zeta(5)] (p^2)^{k-2-(k+1)\epsilon} + \mathcal{O}(1)$$

The momentum integral for the graph 3b) gives

$$\int \frac{d^n q_3 \, \dots \, d^n q_{k-1}}{(2\pi)^{n(k+1)}} \frac{-q_3^2}{(q_4 - q_3)^2 \dots (p - q_{k-1})^2} \times$$
$$\int \frac{d^n k \, d^n l \, d^n r \, d^n s}{k^2 l^2 (k-l)^2 (r-k)^2 (s-l)^2 (r-s)^2 (q_3 - r)^2 (q_3 - s)^2} \tag{A.10}$$
$$= \frac{1}{\epsilon} \left[\frac{1}{(4\pi)^2} \right]^{k+1} \frac{(-1)^k (k-1)}{[(k-1)!]^2 (k+1)} 40\zeta(5) (p^2)^{k-2-(k+1)\epsilon} + \mathcal{O}(1)$$

Finally, the graphs 3c) and 3d) lead to the same contribution

$$\int \frac{d^n r \, d^n q_2 \, \dots \, d^n q_{k-1}}{(2\pi)^{n(k+1)}} \frac{1}{(q_2 - r)^2 (q_3 - q_2)^2 \dots (p - q_{k-1})^2} \times$$
$$\int \frac{d^n k \, d^n l}{k^2 l^2 (k-l)^2 (r-k)^2 (r-l)^2} \tag{A.11}$$
$$= \frac{1}{\epsilon} \left[\frac{1}{(4\pi)^2} \right]^{k+1} \frac{(-1)^k (k-1)}{[(k-1)!]^2 (k+1)} 6\zeta(3) (p^2)^{k-2-(k+1)\epsilon} + \mathcal{O}(1)$$

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